Mixed-state sensitivity of several quantum-information benchmarks

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We investigate an imbalance between the sensitivity of the common state measures—fidelity, trace distance, concurrence, tangle, von Neumann entropy, and linear entropy—when acting on by a depolarizing channel. Further, in this context we explore two classes of two-qubit entangled mixed states. Specifically, we illustrate a sensitivity imbalance between three of these measures for depolarized (i.e., Werner-state-like) nonmaximally entangled and maximally entangled mixed states, noting that the size of the imbalance depends on the state’s tangle and linear entropy.

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I. INTRODUCTION

Because the outcome of most quantum-information protocols hinges on the quality of the initial state, pure maximally entangled states are often the optimal inputs. However, decoherence and dissipation inevitably decrease the purity and entanglement of resource states, yielding partially entangled mixed states. The most common measure used to benchmark a starting state resource is the fidelity [1], as used, e.g., in entanglement purification [2,3] and optimal mixed state teleportation [4]. Likewise, the success of these procedures is often judged using the fidelity of the output state with some target, as is the case, for example, in quantum cloning [5]. Recently it was found that, for the specific case of maximally entangled mixed states [6–8] (MEMS), using the fidelity to compare an experimentally produced state and a target state was a less sensitive way of assessing experimental agreement than comparing the tangle [9,10] and the linear entropies [11] of those states [12]. Because one needs to understand the best way to benchmark states for quantum-information protocols, here we examine the fidelity for more general entangled two-qubit mixed quantum states and note its behavior in relation to the common state measures of linear and von Neumann entropy, tangle and concurrence, and trace distance.

After some general calculations for depolarized states, we consider explicitly two classes of two-qubit entangled states acted on with depolarizing channels: nonmaximally entangled states and maximally entangled mixed states. The effect of a depolarizing channel is to make the states we study similar to the Werner states (an incoherent combination of a pure maximally entangled state and completely mixed state) [13,14], which have been realized with polarized photons [15,16]. These two classes of states were chosen because they allow us to study mixed-state entanglement for states of current interest, and also to understand how these states change under uniform depolarization. Such a uniform depolarization model is applicable to many examples of real experimental decoherence.

II. GENERAL SENSITIVITIES OF MEASURES

Before considering specific examples of entangled mixed states, we examine general sensitivities for several measures using generic depolarized density operators. The depolarized $N$-level system ($N=2$ for a qubit, $N=4$ for two qubits, etc.) is

\[ \rho \rightarrow \rho' = (1 - \epsilon) \rho + \frac{\epsilon}{N} I, \]

where $\epsilon$ is the strength of depolarization.

A. Fidelity

For direct comparison of two mixed states, e.g., $\rho$ and $\rho_0$, for target and perturbed states, respectively, we first discuss the fidelity introduced by Jozsa [1]:

\[ F(\rho, \rho_0) = |\text{Tr}(\sqrt{\rho} \rho_0^{1/2})|^2. \]

In the simpler case of two pure states $|\psi_i\rangle$ and $|\psi_p\rangle$, $F$ reduces to $|\langle\psi_i | \psi_p\rangle|^2$. It is also important to note that some researchers, as in [17], use an amplitude version of the fidelity: $f = \sqrt{F}$. In either case, the fidelity is zero for orthogonal states and 1 for identical states.

Because we wish to consider small perturbations in the fidelity, the “amplitude version” $f$ should be less sensitive because it lacks the square. We consider a generic state $\rho$ with eigenvalues $\{\lambda_i\}$, depolarized by $\epsilon$. The amplitude fidelity $f$ between the output state $\rho'$ and the input $\rho$ is

\[ f(\rho, \rho') = \text{Tr} \sqrt{(1 - \epsilon) \rho^2 + \frac{\epsilon}{N} \rho} \]

\[ = \sum_i \sqrt{(1 - \epsilon) \lambda_i^2 + \frac{\epsilon}{N} \lambda_i}. \]

We assume $\epsilon$ is small such that $\epsilonN=1-N|\lambda|$, where $\lambda$ is the smallest nonzero eigenvalue. Thus, we can expand the above expression to second order in $\epsilon$:

\[ f = \sum_{\lambda_i \neq 0} \lambda_i \left[ 1 + \frac{1}{2} \left( \frac{1-N\lambda}{N\lambda} \right) - \frac{1}{8} \left( \frac{1-N\lambda}{N\lambda} \right)^2 \right] \]

\[ = 1 - \left( \frac{1}{2} - \frac{n_{\text{eff}}}{2N} \right) \epsilon - \sum_{\lambda_i \neq 0} \lambda_i \left( \frac{1-N\lambda}{N\lambda} \right)^2 \epsilon^2 + O(\epsilon^3). \]
is of full rank (i.e., $n_0=N$), the first order term vanishes, and the fidelity is sensitive only to second order in the small depolarizing parameter. If $\rho$ is not full rank, $f$ is sensitive to first order, but becomes less so as the rank becomes higher. Squaring the result (5) in fact gives the same order of sensitivity for $F$.

B. Trace distance

Another possible measure used to compare two states is the trace distance [17], given by

$$D(\rho,\rho') = \frac{1}{2} \text{Tr}|\rho - \rho'|.$$  \hspace{1cm} (6)

Evaluating the trace distance using Eq. (1) gives

$$D(\rho,\rho') = \frac{1}{2} \sum_j \left| \lambda_j - \frac{1}{N} \right| \epsilon.$$  \hspace{1cm} (7)

Here the $1/N$ term comes from the $N \times N$ mixed state (1/N) used to depolarize $\rho$ to create $\rho'$ [Eq. (1)]. Thus, we see that the trace distance is always linearly sensitive to the strength of depolarization, except for $\rho=1/N$, i.e., the fully mixed state. Consequently, the difference between two similar states will in general be less apparent when using $f$ (or $F$) than when using $D$.

C. Linear entropy

To quantify the mixedness of a given state $\rho$, we first consider the linear entropy ($S_L$), which is based on the purity, and for an $N$-level system is

$$S_L(\rho) = \frac{N}{N-1} \left[ 1 - \text{Tr}(\rho^2) \right].$$  \hspace{1cm} (8)

The linear entropy is zero for pure states and $1$ for completely mixed states, i.e., $S_L=1$ for the normalized $N$-qubit identity $1/N$. The change in the linear entropy under a depolarizing channel is

$$\Delta S_L = S_L(\rho') - S_L(\rho) = (2\epsilon - \epsilon^2)(1 - S_L).$$  \hspace{1cm} (9)

Therefore, the linear entropy is always linearly sensitive in $\epsilon$, except when $S_L(\rho)=1$, namely, when $\rho$ is the fully mixed state $1/N$. Thus, the linear entropy is, in general, more sensitive to the depolarizing channel than the fidelity, as was previously shown for the specific case of any depolarized linear pure single-qubit state [18].

D. von Neumann entropy

Another frequently encountered entropy measure is the von Neumann entropy:

$$S_V(\rho) = -\text{Tr}(\rho \ln \rho).$$  \hspace{1cm} (10)

Using Eq. (1) and evaluating $\Delta S = S_V(\rho') - S_V(\rho)$ to first order gives

$$\Delta S = \frac{n_0}{N} \ln \epsilon + \epsilon \left( 1 - S_V(\rho) - \frac{n_0}{N} \right) + \frac{n_0}{N} \ln N - \frac{1}{N} \sum_{\lambda_i = 0} \ln \lambda_i,$$  \hspace{1cm} (11)

where $n_0$ is the number of nonzero (zero) eigenvalues of $\rho$, and $n_0+N=N$. When $\rho$ is not a full rank matrix (i.e., $n_0 \neq 0$), the von Neumann entropy is, to leading order, sensitive in $\epsilon$ in $\epsilon$ (stronger than order $\epsilon$). As the rank becomes higher, this $\epsilon$ in $\epsilon$ sensitivity decreases. When $\rho$ is of full rank (i.e., $n_0=0$ and $n_0=N$), the von Neumann entropy is linearly sensitive in $\epsilon$ unless $S_V=\{(1/N)\Sigma_i \ln \lambda_i$, which is again possible only when $\lambda_i=1/N$, i.e., for the fully mixed state $\rho=1/N$.

E. Concurrence and tangle

Here we examine two ways of quantifying the entanglement of a system, restricting our attention to two-qubit states. We will first derive the variation of the concurrence for an entangled state acted on by a depolarizing channel, then use this to find the result for the tangle, which is the concurrence squared.

1. Concurrence

The concurrence is given by [9]

$$C(\rho) = \max \{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \},$$  \hspace{1cm} (12)

where $\lambda_i$ are the eigenvalues of $\rho \tilde{\rho}$ in nonincreasing order. Here we define $\tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho^* (\sigma_2 \otimes \sigma_2)$ with $\sigma_2 = (0, i).$

Suppose $\{\lambda_i\}$ are arranged in nonincreasing order, and the state $\rho$ is entangled, so that $C(\rho) = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}$. (If $\rho$ is unentangled, $\rho'$, which has added noise, is still unentangled.) To find the concurrence of $\rho'$, we have to evaluate the eigenvalues of the matrix

$$\rho' \tilde{\rho}' = (1-\epsilon)^2 \rho \tilde{\rho} + \frac{\epsilon}{4} (1-\epsilon)(\rho + \tilde{\rho}) + \frac{\epsilon^2}{16}.$$  \hspace{1cm} (13)

We can treat the last two terms as perturbations and evaluate the eigenvalues to leading order:

$$\lambda_i' = (1-\epsilon)^2 \lambda_i + \frac{\epsilon}{4} (1-\epsilon)(\rho + \tilde{\rho}) + \frac{\epsilon^2}{16},$$  \hspace{1cm} (14)

where

$$\langle \rho + \tilde{\rho} \rangle_i = \langle \lambda_i | (\rho + \tilde{\rho}) | \lambda_i \rangle.$$  \hspace{1cm} (15)

For $\epsilon < \lambda$, where $\lambda$ is the smallest nonzero value of $\{\lambda_i\}$, we have, to leading order,

$$\sqrt{\lambda_i'} = (1-\epsilon) \sqrt{\lambda_i} + \frac{\epsilon}{8} \langle \rho + \tilde{\rho} \rangle_i.$$  \hspace{1cm} (16)

Hence, the change in concurrence $[\Delta C = C(\rho') - C(\rho)]$ is given by

$$\Delta C = \frac{n_0}{N} \ln \epsilon + \epsilon \left( 1 - C(\rho) - \frac{n_0}{N} \right) + \frac{n_0}{N} \ln N - \frac{1}{N} \sum_{\lambda_i = 0} \ln \lambda_i.$$  \hspace{1cm} (17)
\[ \Delta C = - \sum_{\lambda_i \neq 0} \frac{\varepsilon}{4} \langle \rho + \tilde{\rho} \rangle_{\lambda_i} + \frac{\varepsilon}{16} - \kappa C(\rho) + \frac{\varepsilon}{8} \frac{\langle \rho + \tilde{\rho} \rangle_{1}}{\sqrt{\lambda_1}} - \sum_{i=2, \lambda_i \neq 0}^{4} \frac{\langle \rho + \tilde{\rho} \rangle_{i}}{\sqrt{\lambda_i}}. \] (17)

The variation of concurrence is thus first order in \( \varepsilon \) except for the unlikely case that

\[ C(\rho) = \frac{\langle \rho + \tilde{\rho} \rangle_{1}}{8 \lambda_1} - \sum_{i=2, \lambda_i \neq 0}^{4} \frac{\langle \rho + \tilde{\rho} \rangle_{i}}{8 \lambda_i}, \] (18)

when \( \rho \) is full rank.

2. Tangle

To characterize a state’s entanglement, one may also use the tangle [9,10], i.e., the concurrence squared:

\[ T(\rho) = C(\rho)^2. \] (19)

Using the result for variation in concurrence, the variation of tangle can now be expressed as \( T'=T=2C\Delta C \). Thus, the tangle is also typically sensitive in the first order to depolarization perturbations.

In summary, we have thus far shown that, under the influence of a small depolarizing channel, the fidelity is not as sensitive as the change in trace distance, linear entropy, von Neumann entropy, concurrence, and tangle. Next we shall illustrate this fact for specific states and investigate the situation for larger depolarization and for variable entanglement.

III. INVESTIGATION FOR SPECIFIC STATES

The first state we consider is similar to the classic Werner state, but we allow arbitrary entanglement through the use of a variable nonmaximally entangled pure state component in addition to the mixed-state dilution:

\[ \rho_1(\varepsilon, \theta) = (1 - \varepsilon)|\Psi(\theta)\rangle\langle\Psi(\theta)| + \frac{\varepsilon}{4} \mathbb{I}_4, \] (20)

with

\[ |\Psi(\theta)\rangle = \cos 2\theta |00\rangle + \sin 2\theta |11\rangle, \] (21)

where the parameter \( \theta \) controls the entanglement and \( \varepsilon \) the mixedness. We choose this parametrization for simplicity and because the entropy and the entanglement of the state are somewhat uncorrelated from each other. In this case, the concurrence is \( C(\rho_1(\varepsilon, \theta)) = \max\{0.2(1 - \varepsilon) \cos 2\theta \sin 2\theta - \varepsilon / 2\} \) (assuming \( \cos 2\theta \sin 2\theta \neq 0 \)) and the linear entropy depends only on \( \varepsilon, S_L(\rho_1(\varepsilon, \theta)) = 2\varepsilon - \varepsilon^2 \). In a similar way, we depolarize a maximally entangled mixed state according to

\[ \rho_2(\varepsilon, r) = (1 - \varepsilon)\rho_{\text{MEMS}}(r) + \frac{\varepsilon}{4} \mathbb{I}_4, \] (22)

where the MEMS, using the parametrizations of concurrence (or equivalently tangle) and linear entropy, is given by [8] the parameter \( r \) is the concurrence of the MEMS.

With these parametrizations, we may map out constant fidelity curves between a target state and a perturbed state in the linear-entropy–tangle plane [we choose these particular measures for calculational simplicity and because Eqs. (20) and (22) cover the entire physically allowed region of the plane]. It is our purpose to use these curves to gain insight as to how the entanglement and mixedness may vary over a constant fidelity curve and how this variation may in turn depend on the amount of entanglement and mixedness. To do this, we calculate the fidelity between a target state \( \rho_1(\varepsilon, \theta) \) and a perturbed state \( \rho_2(\varepsilon, r) \). Specifically, the parameters \( \varepsilon_1 \) and \( \theta_1 \) are varied to create perturbed states of all possible tangle and entropy values as long as the perturbed state has a given fidelity with the target. Likewise, the process is repeated for \( \rho_2(\varepsilon, r) \), but instead varying the parameters of \( \rho_2(\varepsilon_2, r_2) \).

In the pure, maximally entangled limit, both Eqs. (20) and (22) reduce to the maximally entangled state \( |\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \). Therefore, this is a natural state with which to start our discussion. Because Eqs. (20) and (22) occupy different regions of the entropy-tangle plane, it is not surprising that we need to use both equations to map out the constant fidelity curves for \( |\phi^+\rangle \), as shown in Fig. 1. The horizontal curves in the region bounded above by the Werner state curve are traced out by computing the fidelity of \( |\phi^+\rangle \) with Eq. (20). This fidelity is \( F = (1 + \sqrt{7})/2 \) and, surprisingly, does not explicitly depend on the depolarization of the perturbed state. The maximal fidelity of any two-qubit state with maximally entangled pure states was found by Verstraete and Verschelde [19] to be bounded above by \( (1 + \sqrt{7})/2 \). The two-qubit states (20) saturate this bound (as does any two-qubit pure state). Any entangled state that saturates this bound apparently has \( F > 1/2 \), thus allowing concentration of entanglement via the scheme of Bennett et al. [2] without requiring local filtering [20]. Another consequence of this simple fidelity expression is that, when comparing Eq. (20) with \( |\phi^+\rangle \), the fidelity by itself cannot distinguish between pure nonmaximally entangled states and Werner states of the same tangle. For example, both the nonmaximally entangled...
We attribute this to the fact that the fidelity does not provide much insight, so we only present numerical results, yielding the predicted counts one would expect to measure in an experiment if there were no measurement noise or fluctuations. These ideal counts are then perturbed in a statistical way to give a variation one might expect in an experimental measurement for a total collection of ~2000 counts [21]. Note that the sizes and shapes of the simulation and the constant fidelity curves are similar but not identical. As the simulation is random, it behaves somewhat like a depolarizing channel, adding uniform noise (explaining some of the similarity); however, random fluctuations are not enough to mimic the extreme changes along the MEMS curve, as the MEMS density matrices possess a very specific form.

The two previous cases dealt with states that have the highest entanglement values, i.e., they are bounded by the edges of the physically allowed regions of the entropy-tangle plane. To investigate the behavior “on the open plane,” we examine an entangled mixed target state that is a specific example of (20):

\[ \rho_I \approx 0.225, \sim 11.57° \]

\[
\begin{pmatrix}
0.7113 & 0 & 0 & 0.2800 \\
0 & 0.0564 & 0 & 0 \\
0 & 0 & 0.0564 & 0 \\
0.2800 & 0 & 0 & 0.1760 \\
\end{pmatrix}
\]

which is shown as a star in Fig. 3. Note that the 0.99-fidelity region is much larger than for any of the previous target states, including the MEMS. This result is particularly astonishing when viewed in light of what is typically considered
In summary, we have shown an imbalance between the sensitivities of the common state measures—fidelity, trace distance, concurrence, tangle, linear entropy, and von Neumann entropy—for two classes of two-qubit entangled mixed states. This imbalance is surprising in light of the fact that orthogonal states which have zero fidelity with one another may have the same entanglement and mixedness; thus, one might have expected the fidelity to be a more sensitive means to characterize a state than quantifying state properties like entanglement and mixedness. Here we have shown an opposite effect. Specifically, we have investigated several examples at different locations in the entropy-tangle plane, where the trend shows progressively larger 0.99-fidelity regions as the state becomes more mixed and less entangled. We also have shown that, at least for maximally entangled target states, the fidelity is insensitive when comparing between Werner states and nonmaximally entangled states of the same tangle. This work has important ramifications for benchmarking the performance of quantum-information processing systems, as it reveals that the usually quoted measure of fidelity is often a remarkably poor indicator, e.g., of the entanglement in a state, on which the performance of quantum-information systems often depend. This may have consequences, for example, for determining the limits of fault tolerant quantum computation [23], and it may be beneficial to include other benchmarks in addition to/instead of fidelity when characterizing resources needed for various quantum-information protocols.

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[14] Note that for certain entanglement and mixedness parameterizations, the Werner states are the MEMS [T. C. Wei *et al.*, Phys. Rev. A 67, 022110 (2003)].


[21] In more detail, we project the ideal target state into 16 basis vectors, such as \(|00\rangle, |11\rangle, (0+i1)|0\rangle\), etc., to obtain a list of probabilities of given “measurement” outcomes. These probabilities are then multiplied by a constant number simulating an expected average number of counts in a total basis measurement, e.g., what one would expect to observe when projecting into \(|00\rangle, |01\rangle, |10\rangle, \text{ and } |11\rangle\). Next, each of these ideal counts (plus 1 to avoid zero distributions) is used as the mean of a Poisson distribution, from which a random number is generated. These “measurement” values are then processed using a maximum likelihood technique to give a physically valid perturbed density matrix [D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A 64, 052312 (2001); J. B. Altepeter, D. F. V. James, and P. G. Kwiat, Quantum State Estimation, edited by M. Paris and J. Rehacek (Springer, Berlin, 2004), Chap. 3]. If the fidelity between the perturbed density matrix and the target state is greater than 0.9900, the tangle and linear entropy are calculated and plotted in Fig. 2.
