Chapter 3

Symmetric Stress-Energy Tensor

We noticed that Noether's conserved currents are arbitrary up to the addition of a divergence-less field. Exploiting this freedom the canonical stress-energy tensor $\Theta^{\mu\nu}$ can be modified to a new tensor $T^{\mu\nu}$ such that $\partial_\mu T^{\mu\nu} = 0$ and $\int_{M_n} d^n x\, T^{0\nu} = \int_{M_n} d^n x\, \Theta^{0\nu}$ (here M_n denotes the spatial sub-manifold of the space-time M). The second condition guarantees that the new tensor $T^{\mu\nu}$ defines the same physical observable (namely, energy-momentum of the field). From Green's theorem, such a modification of $\Theta^{\mu\nu}$ require the existence of an anti-symmetric Belinfante [1] tensor field $B^{\alpha\mu\nu}(x)$ such that

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu}, \qquad B^{\alpha\mu\nu} = -B^{\mu\alpha\nu} \tag{3.1}$$

In this chapter we will discuss a necessary and sufficient conditions for the existence of a Belinfante tensor such that $T^{\mu\nu}$ is symmetric. Our main goal here is to introduce notations, and to summarize the results necessary to make the thesis self-contained.

3.1 Necessary and Sufficient Conditions

Theorem: The anti-symmetric part of the conserved canonical stress-energy tensor is a total divergence, if and only if there exists a symmetric stress-energy tensor [1]. ■

Proof of Sufficiency: Suppose

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\alpha H^{\alpha\mu\nu} \tag{3.2}$$

By definition, $H^{\alpha\mu\nu} = -H^{\alpha\nu\mu}$. Choose

$$B^{\alpha\mu\nu} = \frac{1}{2}\left(H^{\alpha\mu\nu} + H^{\mu\nu\alpha} - H^{\nu\alpha\mu}\right) \tag{3.3}$$

This tensor have the right anti-symmetry $B^{\alpha\mu\nu} = -B^{\mu\alpha\nu}$, and also

$$B^{\alpha\mu\nu} - B^{\alpha\nu\mu} = H^{\alpha\mu\nu} \tag{3.4}$$

Applying eqns. (3.2) and (3.4) in the definition (3.1), we find

$$T^{\mu\nu} - T^{\nu\mu} = \left(\Theta^{\mu\nu} - \Theta^{\nu\mu}\right) + \partial_\alpha (B^{\alpha\mu\nu} - B^{\alpha\nu\mu}) = 0 \tag{3.5}$$

Hence given $H^{\alpha\mu\nu}$ one can explicitly construct a Belinfante tensor $B^{\alpha\mu\nu}$ such that $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu}$ is symmetric. ■

Proof of Necessity: This is in fact trivial. If there exists a symmetric $T^{\mu\nu}$ then from definition (3.1), $\left(\Theta^{\mu\nu} - \Theta^{\nu\mu}\right) = -\partial_\alpha (B^{\alpha\mu\nu} - B^{\alpha\nu\mu})$, a total divergence. ■

3.2 Construction of Belinfante Tensor

From eqn. (2.19) on angular momentum conservation we have seen that a necessary condition for a translation invariant theory be Lorentz invariant is $\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\alpha \left(\Pi^\alpha \Sigma^{\mu\nu} \varphi\right)$, a total divergence. Therefore as a consequence of eqn. (3.3) a full Poincaré invariant field theory always have the following Belinfante tensor which makes $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu}$ a symmetric stress-energy tensor

$$B^{\alpha\mu\nu} = \frac{1}{2}\left(\Pi^\alpha \Sigma^{\mu\nu} + \Pi^\mu \Sigma^{\nu\alpha} - \Pi^\nu \Sigma^{\alpha\mu}\right)\varphi \tag{3.6}$$

It is important to note that, in general, the choice of symmetric stress-energy tensors is *not* unique. This will be our key to the analysis in chapter 5 to construct an improved tensor, if exists, for the scale invariant field theories.

There is an alternative definition of symmetric stress-energy tensor in general relativity [35]. The functional derivative of the action minimally generalized to a metric compatible Riemannian manifold M_R through the correspondence relations $\left(\eta_{\mu\nu} \to g_{\mu\nu}(x), \partial_\mu \to \nabla_\mu, \nabla_\mu g_{\alpha\beta} = 0, d^d x \to d^d x\sqrt{|g|}\right)$ is defined as a symmetric stress-energy tensor in general relativity. We will show in appendix A that these two symmetric stress-energy tensors are identical in flat space-time [32-34]

$$\frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g_{\alpha\beta}(x)=\eta_{\alpha\beta}} = T_{\mu\nu} \tag{3.7}$$

We wish to express the angular momentum in terms of symmetric stress-energy tensor. Substituting $\Theta^{\mu\nu} = T^{\mu\nu} - \partial_\alpha B^{\alpha\mu\nu}$ in eqn. (2.18), we find

$$x^\lambda T^{\mu\rho} - x^\rho T^{\mu\lambda} = M^{\mu\lambda\rho} + \partial_\alpha F^{\alpha\mu\lambda\rho} \tag{3.8}$$

where $F^{\alpha\mu\lambda\rho} = \left(x^\lambda B^{\alpha\mu\rho} - x^\rho B^{\alpha\mu\lambda}\right)$ is anti-symmetric in (α, μ) and in (λ, ρ). Dropping this anti-symmetric divergence, we obtain the conventional angular momentum tensor

$$J^{\mu\lambda\rho} = x^\lambda T^{\mu\rho} - x^\rho T^{\mu\lambda} \tag{3.9}$$

It defines the same Lorentz generators as $M^{\mu\lambda\rho}$. Due to the symmetry of $T^{\mu\nu}$, the conservation law $\partial_\mu J^{\mu\lambda\rho} = 0$ is now an *identity*.

It is important to note that $T^{\mu\nu}$ and hence $J^{\mu\lambda\rho}$ are gauge independent. The action is invariant under a gauge transformation of the Lagrangian $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \partial_\mu Z^\mu(x)$ such that $Z^\mu(x)\big|_{\partial\Lambda} = 0$. The canonical stress-energy tensor transforms as $\Theta^{\mu\nu} \to \Theta'^{\mu\nu} = \Theta^{\mu\nu} + \left(\partial^\nu Z^\mu - \eta^{\mu\nu}\partial_\alpha Z^\alpha\right)$. Treating $Z^\mu(x)$ as an independent vector field in the Lagrangian, one finds from eqns. (2.15b) and (3.6) that

$B^{\alpha\mu\nu} \rightarrow B'^{\alpha\mu\nu} = B^{\alpha\mu\nu} - \left(\eta^{\alpha\nu} Z^{\mu} - \eta^{\mu\nu} Z^{\alpha}\right)$ which clearly shows the gauge independence of $T^{\mu\nu}$, namely $T'^{\mu\nu} = \Theta'^{\mu\nu} + \partial_\alpha B'^{\alpha\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu} = T^{\mu\nu}$.

3.3 Some Examples

We can now apply the definition (3.6) of Belinfante tensor with the spin-matrices (2.15a,b) and (2.16a,b,c) to construct the symmetric stress-energy tensor for real scalar, vector, and Dirac bi-spinor fields.

3.3.1 Real Scalar Field

Scalar fields are spinless: $\Sigma^{\mu\nu} = 0 \Rightarrow T^{\mu\nu} = \Theta^{\mu\nu}$ and $J^{\mu\lambda\rho} = M^{\mu\lambda\rho}$. These results also follow from a direct calculation using the standard Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \tag{3.10}$$

Explicit calculation shows

$$T^{\mu\nu} = \Theta^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu\nu}\mathcal{L} \tag{3.11a}$$

$$J^{\mu\lambda\rho} = M^{\mu\lambda\rho} = \partial^\mu \varphi\left(x^\lambda \partial^\rho \varphi - x^\rho \partial^\lambda \varphi\right) + \left(\eta^{\mu\lambda} x^\rho - \eta^{\mu\rho} x^\lambda\right)\mathcal{L} \tag{3.11b}$$

It is important to note that the trace of $T^{\mu\nu}$ is

$$T^{\mu}{}_{\mu} = \left(1 - \frac{d}{2}\right)\partial^\mu \varphi \partial_\mu \varphi + dV(\varphi) \tag{3.12}$$

Notice that, if $V(\varphi)=0$, then the trace vanishes identically in $d=2$. This is a consequence of general conformal invariance.

3.3.2 Vector Field

The Lagrangian for a massive free $U(1)$ vector field is

$$\mathcal{L}=-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}+\frac{m^2}{8\pi}A_\alpha A^\alpha \tag{3.13a}$$

$$F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha \tag{3.13b}$$

If $m\neq 0$ the gauge freedom does not exist and the Lorentz gauge condition $\partial_\alpha A^\alpha=0$ is an *ad-hoc* constraint on the vector field $A^\mu(x)$ due to the equation of motion $\partial_\alpha F^{\alpha\beta}+m^2A^\beta=0$. The canonical stress-energy tensor is

$$\Theta^{\mu\nu}=-\frac{1}{4\pi}F^\mu{}_\lambda\partial^\nu A^\lambda-\eta^{\mu\nu}\mathcal{L} \tag{3.14}$$

From the spin-matrices (2.15b), we find the Belinfante tensor for the vector field as: $B^{\alpha\mu\nu}=-\frac{1}{4\pi}F^{\alpha\mu}A^\nu$. Applying the equation of motion, one finds the well-known symmetric stress-energy tensor

$$T^{\mu\nu}=\frac{1}{4\pi}\left(F^\mu{}_\lambda F^{\lambda\nu}+\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)+\frac{m^2}{4\pi}\left(A^\mu A^\nu-\frac{1}{2}\eta^{\mu\nu}A_\alpha A^\alpha\right) \tag{3.15}$$

Notice that the trace of $T^{\mu\nu}$ is

$$T^{\mu}{}_{\mu} = \frac{1}{4\pi}\left(\frac{d}{4}-1\right)F_{\alpha\beta}F^{\alpha\beta} + \frac{m^2}{4\pi}\left(1-\frac{d}{2}\right)A_{\alpha}A^{\alpha} \tag{3.16}$$

If $m=0$ ($U(1)$ gauge theory of photons), the trace of the symmetric stress-energy tensor identically vanishes in $d=4$. This is again a consequence of general conformal invariance.

3.3.3 Dirac Bi-Spinor Field

A real scalar Lagrangian for the free Dirac field is

$$\mathcal{L} = \frac{1}{2}\overline{\psi}\left(i\gamma^{\mu}\vec{\partial}_{\mu} - m\right)\psi - \frac{1}{2}\overline{\psi}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu} + m\right)\psi \tag{3.17}$$

The arrow on top of the differential operator denotes its direction of operation. The canonical stress-energy tensor is

$$\Theta^{\mu\nu} = \frac{i}{2}\overline{\psi}\left(\gamma^{\mu}\vec{\partial}^{\nu} - \gamma^{\mu}\overleftarrow{\partial}^{\nu}\right)\psi \tag{3.18}$$

Recalling the spin-matrices (2.16a,b,c), we obtain the Belinfante tensor $B^{\alpha\mu\nu} = \frac{1}{8}\overline{\psi}\left(\left[\gamma^{\alpha},\sigma^{\mu\nu}\right]_{+} + \left[\gamma^{\mu},\sigma^{\nu\alpha}\right]_{+} - \left[\gamma^{\nu},\sigma^{\alpha\mu}\right]_{+}\right)\psi$. This expression can be further simplified by using the Clifford algebra of Dirac matrices $\left[\gamma^{\mu},\gamma^{\nu}\right]_{+}=2\eta^{\mu\nu}$ and the identity $\left[A,[B,C]_{-}\right]_{+}-\left[B,[C,A]_{-}\right]_{+}=\left[[A,B]_{+},C\right]_{-}$

$$B^{\alpha\mu\nu}=\frac{1}{8}\overline{\psi}\left[\gamma^{\alpha},\sigma^{\mu\nu}\right]_{+}\psi \tag{3.19}$$

Using the commutation relation $\left[\gamma^{\alpha},\sigma^{\mu\nu}\right]_{-}=2i\left(\eta^{\alpha\mu}\gamma^{\nu}-\eta^{\alpha\nu}\gamma^{\mu}\right)$ and then applying the equations of motion $\left(i\gamma^{\mu}\vec{\partial}_{\mu}-m\right)\psi=0=\overline{\psi}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu}+m\right)$, we obtain the symmetric stress-energy tensor for Dirac bi-spinor field

$$T^{\mu\nu}=\frac{i}{4}\overline{\psi}\left(\gamma^{\mu}\vec{\partial}^{\nu}+\gamma^{\nu}\vec{\partial}^{\mu}-\gamma^{\mu}\overleftarrow{\partial}^{\nu}-\gamma^{\nu}\overleftarrow{\partial}^{\mu}\right)\psi \tag{3.20}$$

Straightforward calculation yields the angular momentum tensor as

$$J^{\mu\lambda\rho}=\frac{i}{2}\overline{\psi}\left(\gamma^{\mu}\vec{\partial}^{[\rho}x^{\lambda]}+\gamma^{[\rho}\vec{\partial}^{|\mu|}x^{\lambda]}+x^{[\rho}\overleftarrow{\partial}^{|\mu|}\gamma^{\lambda]}+x^{[\rho}\gamma^{|\mu|}\overleftarrow{\partial}^{\lambda]}\right)\psi \tag{3.21}$$

Here we have used the notation: $Q^{[\alpha|\mu|\beta]}\overset{def}{=}\frac{1}{2}\left(Q^{\alpha\mu\beta}-Q^{\beta\mu\alpha}\right)$. The trace of $T^{\mu\nu}$ is

$$T^{\mu}{}_{\mu}=m\overline{\psi}\psi \tag{3.22}$$

If $m=0$, then it is traceless for all space-time dimensions. Massless Dirac field is general-conformal invariant in all space-time dimensions.