Chapter 2

Space-Time Symmetries

In classical field theory any continuous symmetry of the action generates a conserved current by Noether's procedure. If the Lagrangian is not invariant but only shifts by a divergence the same procedure still applies.

We choose the fields $\varphi^A(x)$ as various representations of Lorentz group in a $d-$dimensional flat space-time manifold M. In an infinitesimal proper orthochronous Lorentz transformation $x^\mu \rightarrow x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$, $|\omega^\mu{}_\nu| << 1$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ the fields transform as

$$\varphi^A(x) \rightarrow \varphi'^A(x') = \left[\delta^A{}_B + \frac{1}{2}\omega_{\mu\nu}\left(\Sigma^{\mu\nu}\right)^A{}_B\right]\varphi^B(x) \tag{2.1}$$

where $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$ are the spin matrices. $\varphi^A(x)$ transforms like a scalar under space-time translation $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$. The action $S[\varphi] = \int\limits_{\Lambda\subset M} d^d x \, \mathcal{L}(\varphi, \partial_\mu\varphi)$ is invariant under the full Poincaré group of Lorentz transformation and space-time translation. Here the domain $\Lambda \subset M$ is an invariant set under the transformation.

2.1 Noether's Theorem

The action of a field theory described by a local Lagrangian $\mathcal{L}\left(\varphi^A(x), \partial_\mu \varphi^A(x)\right)$ is

$$S[\varphi] = \int_{\Lambda \subset M} d^d x \, \mathcal{L}\left(\varphi^A, \partial_\mu \varphi^A\right) \tag{2.2}$$

Suppose under an infinitesimal diffeormorphism $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \left|\varepsilon^\mu(x)\right| << 1$ the field, the Lagrangian, and the action transform as

$$\varphi^A(x) \to \varphi'^A(x) = \varphi^A(x) + \delta\varphi^A(x) \tag{2.3a}$$

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \delta\mathcal{L}(x) \tag{2.3b}$$

$$S[\varphi] \to S'[\varphi'] = S[\varphi] + \delta S[\varphi] \tag{2.3c}$$

where $\mathcal{L}'(x) = \mathcal{L}'\left(\varphi'^A(x), \partial_\mu \varphi'^A(x)\right)$, then up to the first order in $\varepsilon^\mu(x)$

$$\delta S[\varphi] = \int_{\Lambda \subset M} d^d x \left[\delta\mathcal{L} + \partial_\mu\left(\varepsilon^\mu \mathcal{L}\right)\right] \tag{2.4}$$

If $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \left|\varepsilon^\mu(x)\right| << 1$ is a symmetry of the field theory, then $\delta S[\varphi] = 0$. If the choice of the domain $\Lambda \subset M$ is arbitrary, then

$$\delta S[\varphi] = 0 \quad \Rightarrow \quad \delta\mathcal{L} + \partial_\mu\left(\varepsilon^\mu \mathcal{L}\right) = 0 \tag{2.5}$$

This is a very strong condition. In the weakest situation $\Lambda = M$, so $\delta S[\varphi] = 0$ is a necessary and sufficient condition for the existence of a vector field $V^\mu(x)$ such that

$$\delta\mathcal{L} + \partial_\mu\left(\varepsilon^\mu \mathcal{L}\right) = \partial_\mu V^\mu \tag{2.6a}$$

$$V^\mu(x)\Big|_{infinity} = 0 \tag{2.6b}$$

Invariance of the action $S'[\varphi'] = S[\varphi']$ tells us $\mathcal{L}'(x') = \mathcal{L}\left(\varphi'^A(x'), \partial'_\mu \varphi'^A(x')\right)$. Using the equation of motion $\frac{\partial\mathcal{L}}{\partial\varphi^A} - \partial_\mu \Pi^\mu{}_A = 0$, where $\Pi^\mu{}_A = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \varphi^A\right)}$, one finds

$$\delta\mathcal{L}(x) = \partial_\mu\left(\Pi^\mu{}_A \delta\varphi^A(x)\right) \tag{2.7}$$

Substituting this into eqn. (2.6a), we find

$$\partial_\mu\left(\Pi^\mu{}_A \delta\varphi^A + \varepsilon^\mu \mathcal{L} - V^\mu\right) = 0 \tag{2.8}$$

Therefore Noether's conserved current for a space-time symmetry is

$$J^\mu(x) = \Pi^\mu{}_A(x)\delta\varphi^A(x) + \varepsilon^\mu(x)\mathcal{L}(x) - V^\mu(x) \tag{2.9}$$

$J^{\mu}(x)$ is arbitrary up to the addition of a divergence-less field and up to a change of scale and sign. In the subsequent analysis we will use eqn. (2.9) to construct the canonical conserved currents associated with the various space-time symmetries.

2.2 Poincaré Currents

Since the volume element (measure) $d^d x$ is invariant under a proper orthochronous Poincaré transformation $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$, $\left(\det\left(\Lambda^{\mu}{}_{\nu}\right) = +1,\ \Lambda^{0}{}_{0} \geq +1\right)$, the action $S[\varphi]$ is invariant in arbitrary domain $\Lambda \subset M$ when the Lagrangian $\mathcal{L}(x)$ is a Poincaré scalar. Therefore we will apply the strong condition (2.5) for calculating the Poincaré currents.

Canonical stress-energy tensor is the conserved current associated with the $d-$parameter space-time translation group $x^{\mu} \to x'^{\mu} = x^{\mu} + a^{\mu}$. The fields $\varphi^{A}(x)$ form a basis to the various representations of the Lorentz group and transform like scalars under translation

$$\delta\varphi^{A}(x) = \varphi'^{A}(x) - \varphi^{A}(x) = -a^{\nu}\partial_{\nu}\varphi^{A}(x), \qquad \left|a^{\mu}\right| << 1 \tag{2.10}$$

We read the conserved current from eqn. (2.9) as

$$\Theta^{\mu\nu} = \Pi^{\mu}{}_{A}\partial^{\nu}\varphi^{A} - \eta^{\mu\nu}\mathcal{L}, \qquad \partial_{\mu}\Theta^{\mu\nu} = 0 \tag{2.11}$$

This is the well-known canonical stress-energy tensor. Conservation follows *identically* by the equation of motion. The anti-symmetric part of $\Theta^{\mu\nu}$ is given by

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \Pi^{\mu}{}_{A}\partial^{\nu}\varphi^{A} - \Pi^{\nu}{}_{A}\partial^{\mu}\varphi^{A} \tag{2.12}$$

The conserved current associated with the proper orthochronous Lorentz transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu}, \left|\omega^{\mu}{}_{\nu}\right| << 1, \omega_{\mu\nu} = -\omega_{\nu\mu}$ is the canonical angular momentum tensor. The fields $\varphi^{A}(x)$ are representations of Lorentz group which transform like

$$\varphi^{A}(x) \to \varphi'^{A}(x') = \left[\delta^{A}{}_{B} + \frac{1}{2}\omega_{\mu\nu}\left(\Sigma^{\mu\nu}\right)^{A}{}_{B}\right]\varphi^{B}(x) \tag{2.13}$$

Due to the group structure of the Lorentz transformation, the spin-matrices $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$ satisfy the following commutation relation [29]

$$\left[\Sigma^{\mu\nu}, \Sigma^{\lambda\rho}\right]_{-} = \eta^{\mu\rho}\Sigma^{\nu\lambda} - \eta^{\mu\lambda}\Sigma^{\nu\rho} + \eta^{\nu\lambda}\Sigma^{\mu\rho} - \eta^{\nu\rho}\Sigma^{\mu\lambda} \tag{2.14}$$

Spin-matrices for scalar, vector, and second rank tensor fields are given by

$$\Sigma^{\mu\nu} = 0 \tag{2.15a}$$

$$\left(\Sigma^{\mu\nu}\right)^{A}{}_{B} = \eta^{\mu A}\delta^{\nu}{}_{B} - \eta^{\nu A}\delta^{\mu}{}_{B} \tag{2.15b}$$

$$\left(\Sigma^{\mu\nu}\right)^{AB}{}_{CD} = \eta^{\mu A}\delta^{\nu}{}_{C}\delta^{B}{}_{D} - \eta^{\nu A}\delta^{\mu}{}_{C}\delta^{B}{}_{D} + \eta^{\mu B}\delta^{\nu}{}_{D}\delta^{A}{}_{C} - \eta^{\nu B}\delta^{\mu}{}_{D}\delta^{A}{}_{C} \tag{2.15c}$$

These equations can be easily generalized for arbitrary rank tensor fields. The spin-matrices for Dirac bi-spinor fields $\psi(x)$ and $\overline{\psi}(x)$ which satisfy $\left(i\gamma^{\mu}\vec{\partial}^{\mu}-m\right)\psi=0$ and $\overline{\psi}\left(i\gamma^{\mu}\overleftarrow{\partial}_{\mu}+m\right)=0$ (here the arrow on top of the differential operator denotes its direction of operation), respectively, are given by

$$\left(\Sigma^{\mu\nu}\right)^{\psi}{}_{\psi}=\frac{1}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]_{-}=-\frac{i}{2}\sigma^{\mu\nu} \tag{2.16a}$$

$$\left(\Sigma^{\mu\nu}\right)^{\overline{\psi}}{}_{\overline{\psi}}=-\frac{1}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]_{-}=\frac{i}{2}\sigma^{\mu\nu} \tag{2.16b}$$

$$\left(\Sigma^{\mu\nu}\right)^{\psi}{}_{\overline{\psi}}=0=\left(\Sigma^{\mu\nu}\right)^{\overline{\psi}}{}_{\psi} \tag{2.16c}$$

Dirac matrices γ^{μ} satisfy Clifford algebra $\left[\gamma^{\mu},\gamma^{\nu}\right]_{+}=2\eta^{\mu\nu}$, and $\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right]_{-}$.

From now on the field indices A, B, C, ... will be suppressed and all the equations will be interpreted as matrix equations with appropriate summation over the field indices. Then from eqn. (2.13)

$$\delta\varphi(x)=\varphi'(x)-\varphi(x)=\frac{1}{2}\omega_{\lambda\rho}\left[\left(x^{\lambda}\partial^{\rho}\varphi-x^{\rho}\partial^{\lambda}\varphi\right)+\Sigma^{\lambda\rho}\varphi\right] \tag{2.17}$$

Noether's current (2.9) defines the conserved canonical angular momentum tensor as

$$M^{\mu\lambda\rho}=\left(x^{\lambda}\Theta^{\mu\rho}-x^{\rho}\Theta^{\mu\lambda}\right)+\Pi^{\mu}\Sigma^{\lambda\rho}\varphi,\qquad \partial_{\mu}M^{\mu\lambda\rho}=0 \tag{2.18}$$

which is anti-symmetric in (λ,ρ).

The conservation law $\partial_\mu M^{\mu\lambda\rho} = 0$ does *not* follow as an identity for arbitrary Lagrangians. Therefore Lorentz invariance constrains the canonical stress-energy tensor. Using $\partial_\mu \Theta^{\mu\nu} = 0$, we find

$$\partial_\mu M^{\mu\lambda\rho} = 0 \qquad \Rightarrow \qquad \Theta^{\lambda\rho} - \Theta^{\rho\lambda} = -\partial_\mu \left(\Pi^\mu \Sigma^{\lambda\rho} \varphi \right) \tag{2.19}$$

Hence, if a translation invariant field theory is also Lorentz invariant, then the anti-symmetric part of the canonical stress-energy tensor has to be a total divergence. It is interesting to notice that: $\int_\Lambda d^d x \Theta^{\lambda\rho} = \int_\Lambda d^d x \Theta^{\rho\lambda}$ if $\varphi(x)\big|_{\partial\Lambda} = 0$. In the next chapter we will prove the weakest set of necessary and sufficient conditions for the existence of a symmetric stress-energy tensor, and we will develop an algorithm to construct it from the canonical stress-energy tensor over a flat space-time manifold.