Chapter 1

Introduction

Stress-energy tensor is the most important object in a field theory and have been studied extensively [1-6]. In particular, the finiteness of stress-energy tensor has received great attention because of its relevance to physics. A weak gravitational field will couple linearly to the matter field stress-energy tensor and its matrix elements are observables and hence should be finite. Callan, Coleman, and Jackiw proved that the matrix elements of the conventional symmetric stress-energy tensor are cut-off dependent for most renormalizable quantum fields [4]. However, they found that it is always possible to improve the symmetric stress-energy tensor for a renormalizable field theory over (3+1)-dimensional flat space-time manifold. The new tensor defines the same field energy-momentum and angular momentum as the conventional tensor, and further, it has finite matrix elements in every order of renormalized φ^4 perturbation theory [4-7]. This improved tensor is traceless for a non-interacting field theory when all coupling constants are physically dimensionless.

Coleman's improved stress-energy tensor opened a paradigm to analyze wider class of symmetries of a field theory. Local Lagrangian field theories with no dimensional coupling constants might have a larger space-time symmetry group, namely the (d+1)(d+2)/2-parameter conformal group over a d-dimensional flat space-time. The flat space-time conformal group consists of three subgroups, namely d(d+1)/2-parameter Poincaré group, 1-parameter group of scale or dilatation transformation, and d-parameter group of special-conformal transformation. It turns out that scale transformation is always a necessary condition, but in a flat space-time of $d \geq 3$ special conformal symmetry is *only* a strong sufficient condition for the existence of a traceless symmetric improved stress-energy tensor [8-10]. For example, a free massless $U(1)$ gauge theory (that is, sourceless Maxwell's field equations) is fully conformal invariant only in $d = 4$ [11]. However, I will show that an improved traceless stress-energy tensor exists for a free massless $U(1)$ gauge field in (almost) all space-time dimensions provided the gauge symmetry is broken and the Lorentz gauge has been imposed for $d \neq 2, 4$ [12]. In a more general case Deser and Schwimmer considered N-form local gauge theories which are conformally invariant over a critical dimension $d = 2(N+1)$, but allow an improved tensor in every space-time dimension when either the gauge symmetry or the Lorentz invariance is broken [9].

I believe that the physical significance of the improvement term is not yet fully understood. Callan, Coleman, and Jackiw found two contradictory results when the improved stress-energy tensor of a scalar φ^4 matter field was used as the source gravity in a 4-dimensional space-time [4]. One of their result implied the mass of a particle depends on the surrounding gravitational field which is in obvious violation of the equivalence principle. But a second *mathematically equivalent* result showed that the only effect of their new gravitation theory is to change, in a universal way, the strength of the quartic self-interaction which is consistent with the principle of equivalence. However, a more careful study by Ohanian claims that gravitational theory based on the improved stress-energy tensor satisfies both weak and strong principle of equivalence [13]. According to Ohanian, the improvement term (which leads to non-minimal coupling in the scalar matter field Lagrangian) produces a new short range gravitational interaction between the source scalar field and other non-source matters (such as, an electron described by a Dirac field) instead of violating the strong principle of equivalence.

Many other authors studied the interaction of a point like scalar particle with another scalar field [14-17]. Kasper showed that the improvement term introduce a velocity dependent force on the test particle which is, if certain conditions are fulfilled, repulsive in nature [17].

These results are all speculative with no experimental evidence so far. It is only when we consider gravitational interaction that the full physical importance of the stress-energy tensor emerges, for it is the source of gravity to lowest order in the gravitational coupling. We need a full quantum theory of gravitation to understand the physics of improved stress-energy tensor.

The question for existence of an improved tensor for a (1+1)-dimensional Poincaré and scale invariant field theory on a flat space-time manifold has been a long standing problem open for almost 30 years [18]. In this thesis I will present a constructive proof of the necessary and sufficient conditions for the existence of a traceless symmetric improved stress-energy tensor for a local relativistic scale invariant Lagrangian field theory in an arbitrary dimensional flat space-time including $d = 2$ [12].

In $d \geq 3$ conformal invariance does not give much more new information than the scale transformation. But in $d = 2$, the conformal algebra becomes infinite dimensional. This restriction puts significant constraints on the 2-dimensional conformal field theories [18]. Also, it turns out that 2-dimensional conformal invariance give restrictions on the allowed space-time dimensions (d = 10 or 26) [19, 20], space-time supersymmetry [21], and restrictions on possible gauge groups [22, 23] in string theory. The role of improved stress-energy tensor is most significant in a (1+1)-dimensional conformal field theories over a flat space-time manifold [24].

The classical scale, or in general, conformal invariance of a massless field theory with physically dimensionless coupling constants is broken by the quantum corrections coming from the renormalization of the coupling constants which leads to trace anomaly in the improved tensor. However, in $d = 2$, there exists special systems, for example, 2-dimensional non-linear sigma model [25, 26], Thirring model [27, 28], gradient coupling model [28, 29], Liouville field theory [30, 31] etc., where the anomaly at the quantum level disappears and the quantum theory becomes invariant to conformal mapping. Improved tensor is the right object to analyze the physics of such (1+1)-dimensional systems. I will derive the improved tensor for Liouville field theory in chapter 7 intrinsically over a flat space-time.

Callan, Coleman, and Jackiw [4, 5] sketchily constructed the improved stress-energy tensor in a 4-dimensional flat space-time. In this thesis, I will present a general algorithm for explicit construction of the improved tensor from a local Lagrangian over an arbitrary dimensional flat space-time. The algorithm will be applied to various examples, namely to the real scalar fields, Liouville field, the Dirac bi-spinor fields, and to the $U(1)$ gauge field. We will begin with a review of Noether's theorem for coordinate symmetries, and then go through the necessary and sufficient conditions for symmetrization of the canonical stress-energy tensor over a flat space-time [1]. This background discussion will be useful to further modify the symmetric tensor to the traceless and symmetric "improved" tensor. Some open questions will be discussed in the conclusion.

It is usually assumed that a symmetric stress-energy tensor is the functional derivative of the action with respect to the metric tensor when the fields are minimally coupled to gravity (flat space-time $\rightarrow$ corresponding metric compatible Riemannian manifold: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x),\ \partial_\mu \rightarrow \nabla_\mu, \nabla_\mu g_{\alpha\beta} = 0,\ d^d x \rightarrow d^d x\sqrt{|g|}$). A rigorous proof that the above replacement defines the conventional symmetric stress-energy tensor, but not in general the improved tensor is given in appendix A [32-34]. If the field theory can be made general-conformal invariant on a Riemann manifold then the improved tensor is identical with the conventional symmetric tensor [35]. As an example, we will discuss the general-conformal action of a special-conformal scalar field theory and show that its functional variation with respect to the metric tensor defines the improved stress-energy tensor on the flat space-time [5].

The structure of this thesis is as follows. We start with describing a general version of Noether's theorem for continuous space-time symmetries and applying it to Poincaré invariance in chapter 2. In chapter 3, we review Belinfante's (not so well-known) method for symmetrizing a canonical stress-energy tensor using spin of the field theory. In chapter 4 we turn to the flat space-time field theories that are invariant under scale transformation. Our goal is to connect the conservation of dilatation current with the trace of a stress-energy tensor. Callan, Coleman, and Jackiw first achieved it in a (3+1)-

dimensional flat space-time. We present a slightly general version of their result in the first section of chapter 5. Their construction of the improved tensor breaks down (diverges) over (1+1)-dimensional space-time, and it remained an unsolved problem from 1970. In the second section of chapter 5, I present the necessary and sufficient conditions for existence of an improved tensor in (1+1)-dimension, and an explicit algorithm to calculate it intrinsically over a flat space-time. After a brief review of conformal invariance in chapter 6, I will present several examples of calculating the improved tensor in chapter 7. I use the algorithm developed in section 5.2 to calculate the improved tensor for Liouville field intrinsically on (1+1)-dimensional flat space-time, and also I show that a vector field always admits an improved tensor in $d \neq 2, 4$ provided the gauge symmetry is broken. We conclude this thesis in chapter 8 with some problems for further research. A concise derivation of the variational theory of symmetric stress-energy tensor is presented in the appendix A.

1.1 Notation

M is a $d = (n+1)$ dimensional flat space-time manifold endowed with the Lorentz metric $\eta_{\mu\nu} = \mathrm{diag}\left(+1, \underbrace{-1,-1,\ldots,-1}_{n-times}\right)$. M_n is the $n-$dimensional spatial submanifold of M. M_R is a corresponding metric compatible Riemannian manifold $\left(\eta_{\mu\nu} \to g_{\mu\nu}(x), \partial_\mu \to \nabla_\mu, \nabla_\mu g_{\alpha\beta} = 0, d^d x \to d^d x\sqrt{|g|}\right)$ with an affine connection $\Gamma^\mu{}_{\alpha\beta} \overset{def}{=} \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right)$ that defines the covariant derivative as $\nabla_\alpha A^\mu \overset{def}{=} \partial_\alpha A^\mu + \Gamma^\mu{}_{\alpha\beta} A^\beta$. $g = \det\left(g_{\mu\nu}\right)$. Summation convention is applied throughout.

The symbol $\Delta \overset{def}{=} \partial_\mu \partial^\mu$ stands for the *d*-dimensional Laplacian. Since we are interested in conservation laws, all Lagrangians are assumed to be independent of the space-time points, so $\frac{\partial \mathcal{L}}{\partial x^\mu} = 0$. The commutator and anti-commutator are denoted by $[A, B]_- \overset{def}{=} AB - BA$ and $[A, B]_+ \overset{def}{=} AB + BA$, respectively. By definition, $\sigma_+^{\mu\nu} \equiv \sigma^{(\mu\nu)} \overset{def}{=} \frac{1}{2}\left(\sigma^{\mu\nu} + \sigma^{\nu\mu}\right)$ and $\sigma_-^{\mu\nu} \equiv \sigma^{[\mu\nu]} \overset{def}{=} \frac{1}{2}\left(\sigma^{\mu\nu} - \sigma^{\nu\mu}\right)$. We choose $c = \hbar = 1$.