Anomalous Hydrodynamics of Vortex flow in Two-Dimensional Fluid

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Hydrodynamics of incompressible fluid

2D incompressible flows consist of vortices

Hydrodynamics of vortex fluid

Vortices as constituencies of a secondary fluid - **the vortex fluid** (or vortex matter)

- Fast motion: fluid precessing around vortices;
- Slow motion of vortices.
- **What is the hydrodynamics of vortex fluid?**

Euler Hydrodynamics $\Rightarrow$ Anomalous hydrodynamics

Outline

- Hydrodynamics: Incompressible flows in two dimensions
- Search for Conformal Invariance
- Kirchhoff equations
- Onsager ensemble and Random Matrix Theory
- Anomalous forces in hydrodynamics
- Hydrodynamics of vortex flow, superfluids and FQHE

Hydrodynamics of vortex flow

Hydrodynamics of the vortex flow is anomalous

Assumption: Circulations of vortices are bounded $> \Gamma$

Anomalous Hydrodynamics

Euler Equation $\boxed{D_t \mathrm{u} = -\nabla p}$

Anomalous Euler Equation $\boxed{D_t \mathrm{v}_a = -\nabla_a p + \nabla_b \tau_{ab}}$

$D_t \equiv \partial_t + \mathbf{u} \cdot \nabla$ - Material Derivative.

τ_{ab} – anomalous stress - symmetric pseudo-tensor

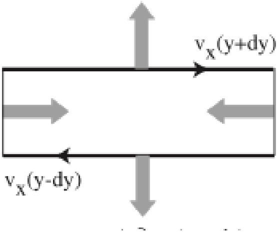
$$2D: \quad \tau_{xy} = \tau_{yx} = -\eta(\nabla_x u_x - \nabla_y u_y),$$

$$\tau_{xx} = -\tau_{yy} = \eta(\nabla_x u_y + \nabla_y u_x),$$

$$\boxed{\tau = \tau_{xx} - \tau_{yy} - 2i\tau_{xy} = -2i\eta\partial u, \quad \tau_{z\bar{z}} = 0}$$

η – a universal anomalous kinetic coefficient

Anomalous Hydrodynamics



Hydrodynamics of incompressible forces

Euler Equation	$D_t u = -\nabla p,$
Material Derivative	$D_t \equiv (\partial_t + u \cdot \nabla)$

Incompressibility	$\nabla \cdot \mathbf{u} = 0,$
Vorticity	$\varpi = \nabla \times \mathbf{u}$

Vorticity is transported along the velocity field: the material derivative of the vorticity in that flow vanishes:

$$\text{Helmholtz Equation}: \qquad \frac{D\varpi}{Dt} \equiv \dot{\varpi} + \mathbf{u} \cdot \nabla \varpi = 0.$$

Kirchhoff Equations

$$\frac{D\varpi}{Dt} \equiv \dot{\varpi} + \mathbf{u} \cdot \nabla \varpi = 0.$$

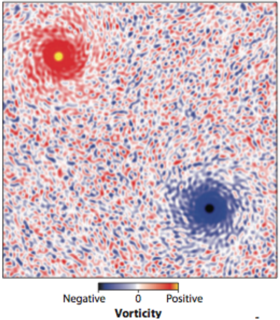
Helmholtz (and later Kirchhoff)

$$\mathrm{u}(z,t) = u_x - \mathrm{i}u_y = \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma_i}{z - z_i(t)}$$

Kirchhoff equations

$$\mathrm{i}\,\dot{\bar{z}}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}$$

Chiral Flow: Clustering, Rotating Fluid



Chiral Kirchhoff equations $\Gamma_i = \Gamma$

$$\mathrm{i}\dot{\bar{z}}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

Object of interest: Large N limit such that the area of the patch is fixed

Canonical structure of Kirchhoff equations

Kirchhoff equations are Hamiltonian

- Poisson brackets

$$\{z_i, \bar{z}_j\}_{P.B.} = (\mathrm{i}\pi\Gamma)^{-1}\delta_{ij}.$$

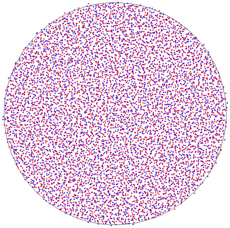
- Hamiltonian

$$\mathscr{H} = \Omega|z_i|^2 - \Gamma^2 \sum_{j\neq i} \log|z_i - z_j|^2$$

Onsager ensemble: Stochastic Hydrodynamics

Thermodynamics of the vortex gas

$$\mathscr{P}(z_1, \ldots, z_N) = \prod_{i \neq j}^{N} |z_i - z_j|^{2\beta} e^{-\sum_i |z_i|^2/4\ell^2}, \quad 2\beta = \Gamma^2/T$$



Hydrodynamics of Onsager fluid of vortices

- Start from the many body system

$$\mathrm{i}\dot{\bar{z}}_i = \mathrm{v}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

- Reformulate it through the density

$$\rho(r) = \sum_i \delta(r - r_i) = (2\pi\Gamma)^{-1} \omega(r).$$

and velocity

$$J = \rho(r)\mathrm{v}(r) \equiv \sum_i \delta(r - r_i)\mathrm{v}_i,$$

- write evolution equations for density ρ and velocity v

$$\mathcal{D}_t \rho = \ldots, \quad \mathcal{D}_t \mathrm{v} = \ldots$$

- Compare with the Euler equations

$$D_t \rho = 0, \quad D_t \mathrm{u} = -\nabla p$$

Chiral relation

In the chiral flow position of vortices determines their velocity:

$$\mathrm{i}\dot{\bar{z}}_i = \mathrm{v}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)} = \frac{1}{\mathrm{i}\pi\Gamma} \partial_{z_i} \mathcal{H}$$

$$\rho \longleftrightarrow \mathrm{v}$$

$$\mathrm{v}(z) = \frac{1}{\mathrm{i}\pi\Gamma} \partial_z \frac{\delta \mathcal{H}}{\delta \rho(z)}$$

Objects in Hydrodynamics

- Flux $\qquad J = \rho\,\mathrm{v} = (\pi\Gamma)\rho\mathrm{v} = \mathrm{i}\bar{\partial}\mathscr{T}$

- Stress tensor: a response of the energy to a general transformation of coordinates and dilatations

$$z \to z + \epsilon(z,\bar{z}), \qquad \text{and dilatations} \quad \rho \to \rho + \lambda(z,\bar{z})\rho$$

$$\mathscr{T}_{z\bar{z}}(z) = -\rho(z)\frac{\delta\mathscr{H}}{\delta\rho(z)},$$

$$\mathscr{T}(z) = \frac{1}{\pi}\sum_i \frac{1}{z - z_i}\frac{\partial\mathscr{H}}{\partial z_i} = \mathrm{i}\Gamma\sum_i \frac{\mathrm{v}_i}{z - z_i}.$$

$$\bar{\partial}\mathscr{T} + \rho\partial(\rho^{-1}\mathscr{T}_{z\bar{z}}) = 0$$

Stress tensor in Anomalous Hydrodynamics

Stress tensor in Euler hydrodynamics (the holomorphic component only):

$$\mathcal{T} = \frac{1}{2}\mathrm{u}^2 = \frac{1}{2}(\partial\psi)^2$$

Stress tensor in vortex (anomalous) hydrodynamics :

$$\mathcal{T} = \frac{1}{2}\mathrm{u}^2 - \mathrm{i}\frac{\Gamma}{2}\partial\mathrm{u} = \frac{1}{2}(\partial\psi)^2 - \underbrace{\mathrm{i}\frac{\Gamma}{2}\partial^2\psi}_{\text{anomalous term}}$$

$$\psi - \text{stream function : } u_x = -\nabla_y\psi,\, u_y = \nabla_x\psi$$

Calculations

We want to express

$$\mathscr{T}(z) = \mathrm{i}\Gamma \sum_i \frac{\mathrm{v}_i}{z - z_i}, \quad \mathrm{v}_i = \sum_{i \neq j}^{N} \frac{\Gamma}{z_i - z_j}$$

Through velocity

$$\mathrm{u} = \sum_j^N \frac{\Gamma}{z - z_j}$$

Use the identity

$$2 \sum_{i \neq j} \frac{1}{z - z_i} \frac{1}{z_i - z_j} = \left(\sum_i \frac{1}{z - z_i} \right)^2 + \partial \left(\sum_i \frac{1}{z - z_i} \right)$$

To obtain

$$\mathscr{T} = \frac{1}{2} \mathrm{u}^2 - \mathrm{i} \frac{\Gamma}{2} \partial \mathrm{u}$$

Deflection of the velocity and stream lines

Anomalous term in the velocity

$$\rho(r)\,\mathrm{v}(r) = \sum_i \delta(r - r_i)\mathrm{v}_i, \quad \mathrm{v}_i = \sum_{i \neq j}^{N} \frac{\Gamma}{z_i - z_j}$$

$$\boxed{\mathrm{v} = \mathrm{u} + \tfrac{\Gamma}{2}\mathrm{i}\partial \log|\omega|}$$

$$\omega = \nabla \times \mathrm{u}$$

and in terms of the stream lines

$$\boxed{\Psi = \psi + \tfrac{\Gamma}{4} \log \Delta\psi}$$

$$\mathrm{v} = -\nabla \times \Psi, \quad \mathrm{u} = -\nabla \times \psi$$

Hamiltonian and Poisson algebra of the vortex flow

$$\mathcal{H} = \frac{1}{2}\int [\mathrm{v}^2 - (\frac{\Gamma}{4}\nabla\log\rho)^2]d^2r,$$

The chiral constraint:

$$(2\pi\Gamma)\cdot(\nabla\times\mathrm{v}) = \rho + \frac{\Gamma}{4}\Delta\log\rho.$$

Poisson algebra $J = \rho\mathrm{v}$

$$\{\rho(r),\,\rho(r')\} = -\pi\Gamma(\nabla_r\times\nabla_{r'})[(\rho(r)+\rho(r'))\delta(r-r')],$$

$$\{\bar{J}(r), J(r')\} = \left(-\frac{1}{2}(J\times\nabla) + \pi\Gamma\left(\rho^2 + \frac{1}{4}\nabla\rho\cdot\nabla\right)\right)\rho\,\delta(r-r')$$

Hydrodynamics in a curved space: Trace anomaly

Riemann manifold with a metric g_{ab}: $\rho \to \rho\sqrt{g}$ The energy

$$H \to H + \underbrace{\frac{\Gamma^2}{32}\int R\,\Delta_g^{-1} R\, dV}_{\text{Liouville action}}$$

The stress tensor

$$\mathcal{T}_{z\bar{z}} \to \mathcal{T}_{z\bar{z}} - \frac{\Gamma^2}{4}R$$

Density in a stationary flow

$$\delta\rho = \frac{1}{8\pi}R$$

A cone with a deficit angle θ accumulates $\theta/4\pi$ vortices.

Kirchhoff Equations are Hamiltonian and finite dimensional - readily for quantization;

$$\mathrm{i}\,\dot{\bar{z}}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}$$

$$\{z_i, \bar{z}_j\}_{P.B.} \to [z_i, \bar{z}_j] = \beta^{-1}\delta_{ij}, \quad \Gamma = \beta\hbar.$$

$$\bar{z}_i \to \beta^{-1}\partial_{z_i}$$

Attempts to quantize incompressible Euler's equation failed.

Relation to FQHE

The ground state of the vortex flow is Feynman wave function.

In the chiral case (all vortices are like-sign) is Laughlin's wave function

$$\mathrm{v}_i|0>=0$$

$$\left(\partial_{z_i}+\Omega\bar{z}_i-\sum_{i\neq j}^{N}\frac{\beta}{z_i-z_j}\right)\Psi(z_1,\ldots,z_N)=0$$

Omitted subjects

- Relation to CFT
- Application to turbulence (project with Eldad)