Holography and Tensor Networks

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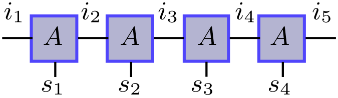
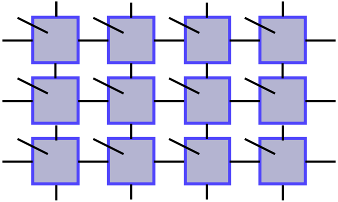
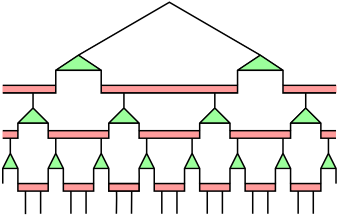
Tensor network wave functions of various kinds:

MPS (matrix product state) or DMRG



PEPS (projected entangled pair state)





MERA
(multiscale entanglement
renormalization ansatz)

Tensor network approach to quantum manybody systems

- Representing many-body wavefunctions by contracting many tensors
  DMRG, MPS, MERA, PEPS, etc.

$$|\Psi\rangle = \sum_{s_1, s_2, s_3, s_4 \cdots} C^{s_1, s_2, s_3, s_4 \cdots} |s_1, s_2, s_3, s_4 \ldots\rangle$$

product state:

$$|\Psi\rangle = \sum_{\{s_a\}} A^{s_1} A^{s_2} A^{s_3} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle = \prod_i \sum_{s_i} A^{s_i} |s_i\rangle$$

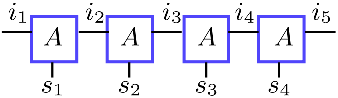
EE = 0

physical degrees of freedom

MPS (matrix product state) :

$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n = 1, \cdots, \chi\}} A^{s_1}_{i_1, i_2} A^{s_2}_{i_2, i_3} A^{s_3}_{i_3, i_4} A^{s_4}_{i_4, i_5} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle$$

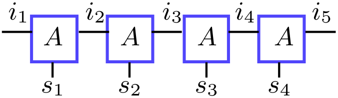
auxiliary index



structure of tensor-network and entanglement entropy

matrix product state (DMRG):

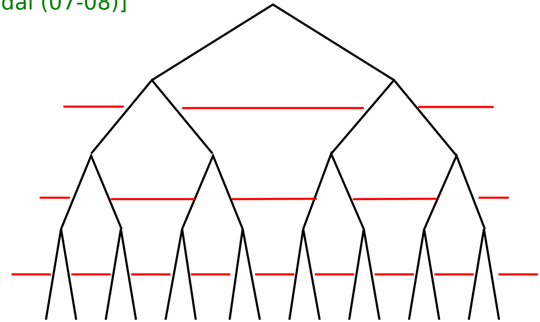
$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2} A^{s_2}_{i_2,i_3} A^{s_3}_{i_3,i_4} A^{s_4}_{i_4,i_5} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle$$

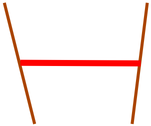


EE: $S_A \leq \log \chi$

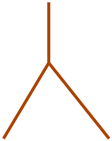
multiscale entanglement renormalization ansatz (MERA)

[Vidal (07-08)]





"disentangler"



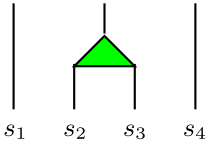
isometry
(coarse-graining)

block spin decimation and disentangler

- Block spin decimation

$$\rho_{tot} = |\Psi\rangle\langle\Psi|$$

$$\rho_{23} = \mathrm{Tr}_{14}\rho_{tot} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$$



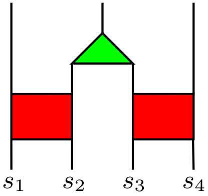
small p_i --> throw away

- Disentangler

$$|\Psi\rangle = \frac{(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)_{12}}{\sqrt{2}}\frac{(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)_{34}}{\sqrt{2}} \qquad p_i = \frac{1}{4}(\forall i)$$

$$U\frac{(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)_{12}}{\sqrt{2}} = |\uparrow\uparrow\rangle_{12}$$

$$\mathrm{Tr}_{14}\left[U_{12}\otimes U_{34}\rho_{tot}(U_{12}\otimes U_{34})^\dagger\right]$$



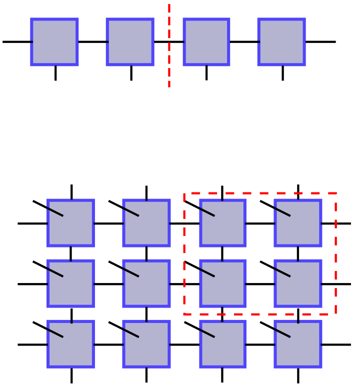
Extracting information from tensor networks

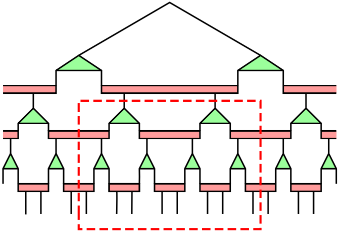
- Entanglement entropy scaling : the best method to measure the central charge (most important parameter in 1D critical system).

- Can we extract information from tensor network in a more effeicent way ?

Basic strategy

Cutting up tensor networks!

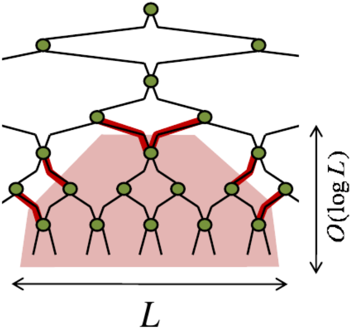




Cutting up = defining "reduced density matrix"

MERA and holographich entanglement entropy

[Swingle (09)]

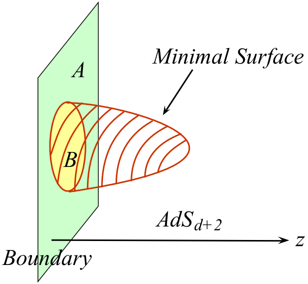


EE: $S_A \sim \log(l/a)$

Geometry <--> Entanglement

- Holographic formula for EE

$$S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}$$



- Entanglement --> geometry

$$ds^2 = g_{uu}du^2 + \frac{e^{2u}}{\epsilon^2}d\vec{x}^2 + g_{tt}dt^2$$

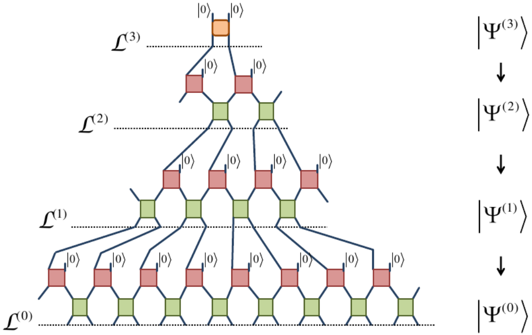
$$S_A = \frac{1}{4G_N} \cdot \frac{V_{d-1}}{\epsilon^{d-1}} \int_{u_{\rm IR}(=-\infty)}^{u_{\rm UV}(=0)} du \sqrt{g_{uu}} e^{(d-1)u}$$

Can we make "AdS/MERA" more precise ?

MERA and quantum circuit

- Tensor network method can be formulated as a quantum circuit (successive applications of unitary transformations)

- For MERA: add dummy states |0>



continuous MERA (cMERA)

[Haegeman-Osborne-Verschelde-Verstraete (11)]

- Quantum circuit representation of the target states:

$$|\Psi(u_{\rm IR})\rangle \equiv |\Omega\rangle \qquad\qquad |\Psi(u_{\rm UV})\rangle \equiv |\Psi\rangle$$

$$|\Psi(u)\rangle = U(u, u_{\rm IR})|\Omega\rangle$$

$$|\Psi\rangle = U(0, u)|\Psi(u)\rangle$$

$$u_{\rm UV} = 0$$

$$u_{\rm IR} = -\infty$$

- MERA evolution operator

$$U(u_1, u_2) = P\exp\left[-i\int_{u_2}^{u_1}(K(u) + L)du\right]$$

disentangler ($K(u)$) coarse-graining (L)

- Optimizing |Omega>, U --> true ground state

cMERA for free boson

- free boson in d+1 dim:

$$H = \frac{1}{2}\int d^dk\,[\pi(k)\pi(-k) + \epsilon_k^2 \cdot \phi(k)\phi(-k)]$$

$$\phi(k) = \frac{a_k + a^\dagger_{-k}}{\sqrt{2\epsilon_k}} \qquad \pi(k) = \sqrt{2\epsilon_k}\left(\frac{a_k - a^\dagger_{-k}}{2i}\right)$$

- IR state:

$$\left(\sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x)\right)|\Omega\rangle = 0$$

completely uncorrelated

$$(\alpha_k a_k + \beta_k a^\dagger_{-k})|\Omega\rangle = 0$$

$$\alpha_k = \frac{1}{2}\left(\sqrt{\frac{M}{\epsilon_k}} + \sqrt{\frac{\epsilon_k}{M}}\right) \qquad \beta_k = \frac{1}{2}\left(\sqrt{\frac{M}{\epsilon_k}} - \sqrt{\frac{\epsilon_k}{M}}\right)$$

- coarse-graining

$$e^{-iuL}\phi(k)e^{iuL} = e^{-\frac{d}{2}u}\phi(e^{-u}k)$$

$$e^{-iuL}\pi(k)e^{iuL} = e^{-\frac{d}{2}u}\pi(e^{-u}k)$$

- disentangler

$$K(u) = \frac{1}{2}\int d^dk\,[g(k,u)(\phi(k)\pi(-k) + \pi(k)\phi(-k))]$$

$$g(k,u) = \chi(u)\cdot\Gamma\left(|k|/\Lambda\right)$$

cutoff function

- variational principle:

$$E = \langle\Psi|H|\Psi\rangle = \langle\Omega|H(u_{\mathrm{IR}})|\Omega\rangle$$

$$\chi(u) = \frac{1}{2}\cdot\frac{e^{2u}}{e^{2u} + m^2/\Lambda^2}, \quad M = \sqrt{\Lambda^2 + m^2}$$

- scale-dependent Bogoliubov transformation:

$$(\alpha_k, \beta_k) \cdot \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} |\Omega\rangle = 0$$

IR

$$U(u)(\alpha_k, \beta_k) \cdot \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} |\Omega\rangle = 0$$

$$(\alpha_k, \beta_k) \cdot U(u) \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} U^\dagger(u) \cdot U(u)|\Omega\rangle = 0$$

$$(\alpha_k, \beta_k) \cdot M(u) \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} |\Psi(u)\rangle_L = 0$$

$$(A_k, B_k) \cdot \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} |\Psi(u)\rangle_L = 0$$

UV

Bures distance (quantum distance)

- Bures distance:

$$D_{\mathrm{B}}(\rho_1, \rho_2) := 2\left(1 - \mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right)$$

- for pure states: $\rho_1 = |\psi_1\rangle\langle\psi_1| \qquad \rho_2 = |\psi_2\rangle\langle\psi_2|$

$$D_{\mathrm{B}}(\psi_1, \psi_2) = 2\left(1 - |\langle\psi_1|\psi_2\rangle|\right)$$

- for infinitesimally close state:

$$D_{\mathrm{B}}[\psi(\xi), \psi(\xi + d\xi)] = g_{ij}(\xi) d\xi_i d\xi_j$$

$$g_{ij}(\xi) = \mathrm{Re}\,\langle\partial_i\psi(\xi)|\partial_j\psi(\xi)\rangle - \langle\partial_i\psi(\xi)|\psi\rangle\langle\psi|\partial_j\psi(\xi)\rangle$$

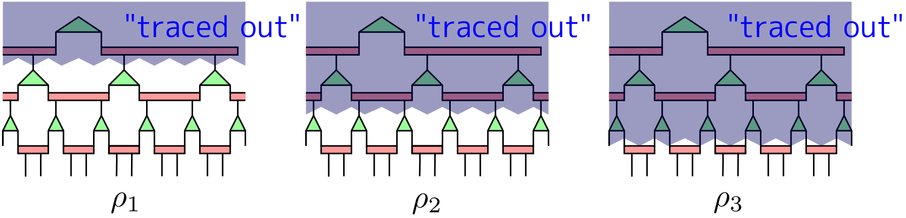
Introducing metric in MERA

- Proposal for a metric in radial direction:

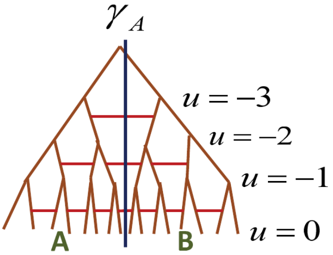
$$g_{uu}(u)du^2 = \mathcal{N}^{-1}\left(1 - |{}_L\langle\Psi(u)|\Psi(u+du)\rangle_L|^2\right)$$

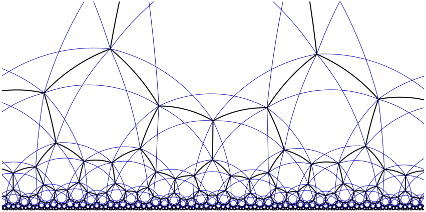
where $|\Psi(u+du)\rangle_L = e^{iLu}|\Psi(u)\rangle$ wfn in "interaction picture"

$\mathcal{N} = \text{Vol.} \int_{|k|\leq\Lambda e^u} d^dk$ normalization



Motivation for the metric





$$S_A \propto L^{d-1} \sum_{u=-\infty}^{0} n(u) \cdot 2^{(d-1)u}$$

strength of disentangler

$$S_A = \frac{1}{4G_N} \cdot \frac{V_{d-1}}{\epsilon^{d-1}} \int_{u_{\mathrm{IR}}(=-\infty)}^{u_{\mathrm{UV}}(=0)} du \sqrt{g_{uu}} e^{(d-1)u}$$

- Relativistic free scalar:

$$ds^2 = g_{uu}du^2 + \frac{e^{2u}}{\epsilon^2}d\vec{x}^2 + g_{tt}dt^2$$

$$g_{uu}(u) = \chi(u)^2 = \frac{e^{4u}}{4(e^{2u} + m^2/\Lambda^2)^2}$$

massless limit:

$$g_{uu}(u) = \text{const.}$$

AdS metric

massive case:

$$e^{2u} = \frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}$$

AdS soliton

$$ds^2 = \frac{dz^2}{4z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right) dx^2 + g_{tt}dt^2$$

- Flat space:

$$H = \int d^d x\, \phi(x) e^{A(-\partial^2)^{w/2}} \phi(x)$$

$$\epsilon_k \propto e^{A \cdot k^w}$$

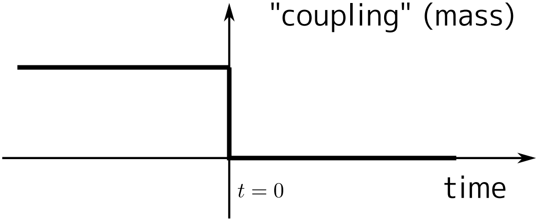
$$g_{uu}(u) = g(u)^2 \propto e^{2wu}$$

c.f. Li-Takayanagi (10)

Quantum quench and finite T

- MERA at finite T (mixed state)

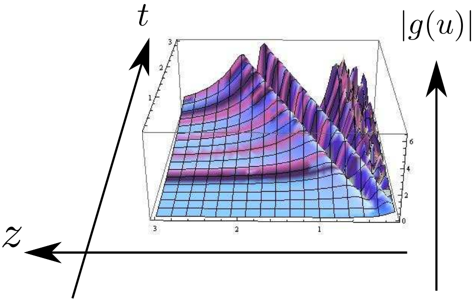
- Quantum quench (pure state)

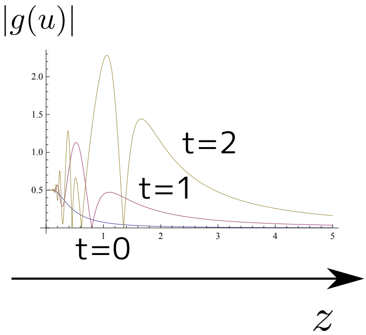


$|\Psi(t=0)\rangle = e^{-\frac{\beta}{4}H}|B\rangle$ "Boundary state" [Calabrese-Cardy 06]

$$|\Psi(t=0)\rangle \propto \exp\left(-\frac{1}{2}\int dk\, e^{-\frac{\beta\varepsilon_k}{2}} a_k^\dagger a_{-k}^\dagger\right)|0\rangle$$

Metric after quantum quench for 2d free boson





- Light-cone like structure

$$g_{uu} = g(u)^2 \simeq \frac{1}{4}\left[1 + \frac{a_1 k^2 \beta^2 + a_2 k^2 t^2}{\sinh^2(k\beta/2)}\right]$$

- t-linear growth of S_A

$$\Delta S_A \sim \int_{-\log\beta/\epsilon}^{0} du \left(\sqrt{g_{uu}} - 1/2\right) + \int_{-\infty}^{-\log\beta/\epsilon} du \sqrt{g_{uu}} \sim \frac{t}{\beta}$$

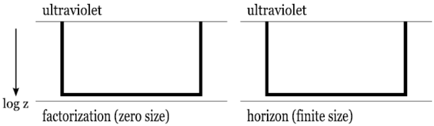
[Calabrese-Cardy (05)
Hartman-Maldacena (13)]

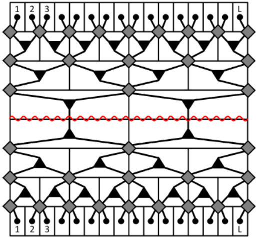
What can we say about finite T ?

- Thermofield double description:

[Matsueda, Molina-Vilaplana, etc]

[Swingle (12)]





- Concrete setup in cMERA (cf. Hartman-Maldacena, next slide):

$$
\begin{aligned}
|\Psi(0,t)\rangle_{th} &= \mathcal{N} \cdot e^{-it(H_1+H_2)} \cdot \prod_k \sum_{n_k=0}^{\infty} e^{-\beta\epsilon_k n_k/2} |n_k\rangle_1 |n_k\rangle_2 \\
&= \mathcal{N} \cdot \exp\left(\int dk e^{-\frac{\beta\epsilon_k}{2}} e^{-2i\epsilon_k t} a_k^\dagger \tilde{a}_k^\dagger \right) |0\rangle|\tilde{0}\rangle.
\end{aligned}
$$

- Can use the same disentangler for quantum quench

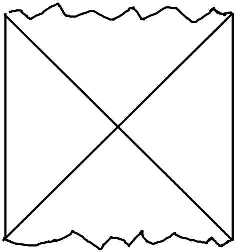
$$
\tilde{a}_k \to a_{-k}, \quad |0\rangle|\tilde{0}\rangle \to |0\rangle.
$$

$$
|\Psi(0,t=0)\rangle_{th} = Pe^{-i\int_{u_{IR}}^{0} \hat{K}(s)ds} \otimes Pe^{-i\int_{u_{IR}}^{0} \hat{\tilde{K}}(\tilde{s})d\tilde{s}} |\Omega(\beta)\rangle.
$$

same metric as quench

Hartman-Maldacena

CFT2

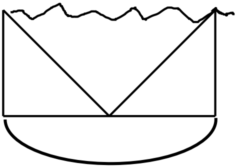


CFT1

$$ds^2 = -\frac{1-z^2/z_H^2}{z^2} d\tau^2 + \frac{dz^2}{z^2(1-z^2/z_H^2)} + \frac{dx^2}{z^2}$$

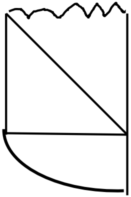
[Israel (76), Maldacena]

$$|\Psi\rangle = \sum_n e^{-\beta E_n/2} |n\rangle_1 |n\rangle_2$$



thermofield double

$$|\Psi\rangle = \sum_n e^{iE_n t} e^{-\beta E_n/2} |n\rangle_1 |n\rangle_2$$



quantum quench

$$|\Psi\rangle = \sum_n e^{iE_n t} e^{-\beta E_n/2} |n\rangle$$

Issues

- Large-N ? higher spin ? [cf. Swingle (12)]
- Diffeo invariance ?

- Time-component of metric g_tt ?
- Effects of interactions ?
- Einstein equation?

[cf. Faulkner-Guica-Hartman-Myers-Van Raamsdonk 13,
Nozaki-Numasawa-Prudenziati-Takayanagi 13,
Bhattacharya-Takayanagi 13, etc]

Advantages of AdS/MERA:

- No need for large-N
- Can define geometry for generic many-body states

Time-dependence

- Time-dependent excited state: $(A_k a_k + B_k a^\dagger_{-k})|\Psi_{ex}\rangle = 0$

$$A_k = \sqrt{1 + a_k^2} \cdot e^{i\epsilon_k t + i\theta_k},$$

$$B_k = a_k \cdot e^{-i\epsilon_k t + i\theta_k},$$

- Phase ambiguirty -- Psi is indep. of theta, but metric is not

- Choose theta s. t. the extremal surface is on the corresponding time-slice.

Topological phases

- Closing comment: topological phases

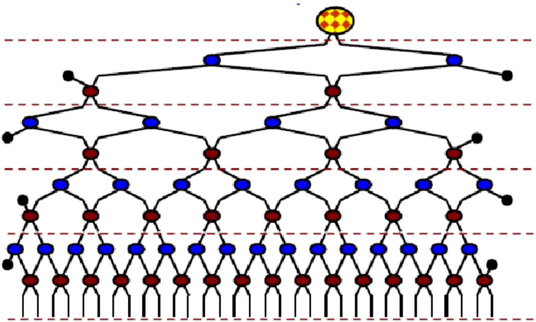
  e.g. phases in condensed matter that are described by
  the Chern-Simons theory

- no classical order parameter,
  highly entangled quantum states of matter

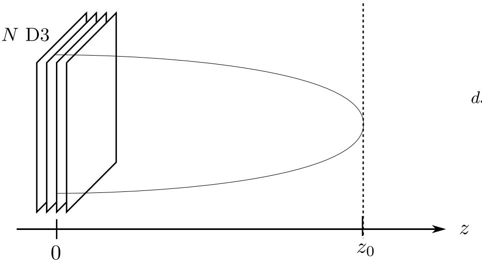
MERA for topological phase

Topological infomation is strored in "top tensor"

[Koenig, Reichardt, Vidal (08)]
[Aguado-Vidal (09)]



AdS/CFT, AdS/CS

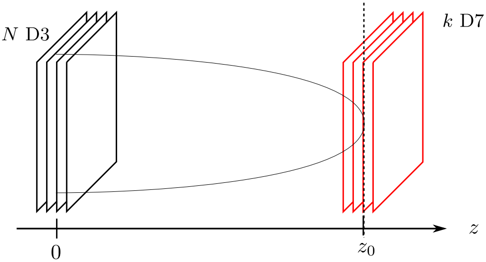


holographic dual of pure YM in (2+1)D

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + f(z)dy^2 + dx_1^2 + dx_2^2 + f^{-1}(z)dz^2\right)$$

$$f(z) = 1 - (z/z_0)^4$$

[Witten (98)]



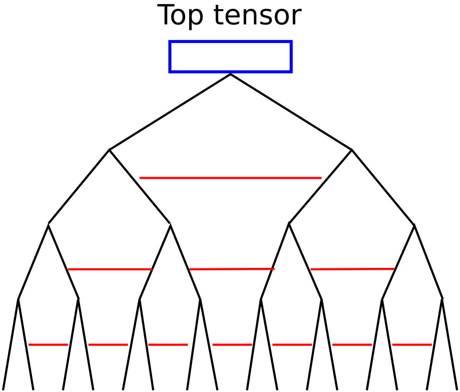
holographic dual of FQHE

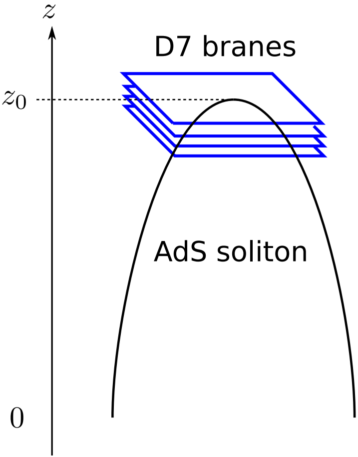
$$S_{\mathrm{D3}} = \frac{k}{4\pi}\int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$$

$$S_{top} \sim \frac{k^2}{2}\log N$$

[Fujita-Li-Ryu-Takayanagi (10)]

D7-brane = "Top tensor" ?





summary and outlook

- Metric for tensor MERA representation of quantum states

- Behaviors expected from AdS/CFT

- Classical phases of matter <-- group theory (symmetry breaking)
  Quantum phases of matter <-- geometry (entanglement)
  Topological phases <--> D-branes