(ML, Z. Gu, 2012)
(C. Wang, ML, in preparation)
(C.-H. Lin, C. Wang, ML, in preparation)

Braiding statistics and symmetry-protected topological phases

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Definition of SPT phases

Gapped quantum many-body system with:

- Some set of symmetries, but no symmetry breaking
- No fractional statistics (“short-range entangled”)
- Cannot be smoothly connected to a “trivial” state with same symmetry
- Gapless symmetry-protected boundary modes

Examples

- Topological insulators (2D/3D, U(1) $\times$ TRS)

  (Hasan, Kane, RMP, 2010)

- Haldane spin-1 chain (1D, TRS)

  (Haldane, 1983)

- Many others…

Basic questions about SPT phases

- **Classification**: For each symmetry group and spatial dimension, how many SPT phases are there?

  - Non-interacting fermions (Schnyder et al, Kitaev, 2008)

  - General boson systems (Chen, Gu, Liu, Wen, 2011)

- **Characterization**: How can we determine whether a microscopic model belongs to a specific SPT phase?

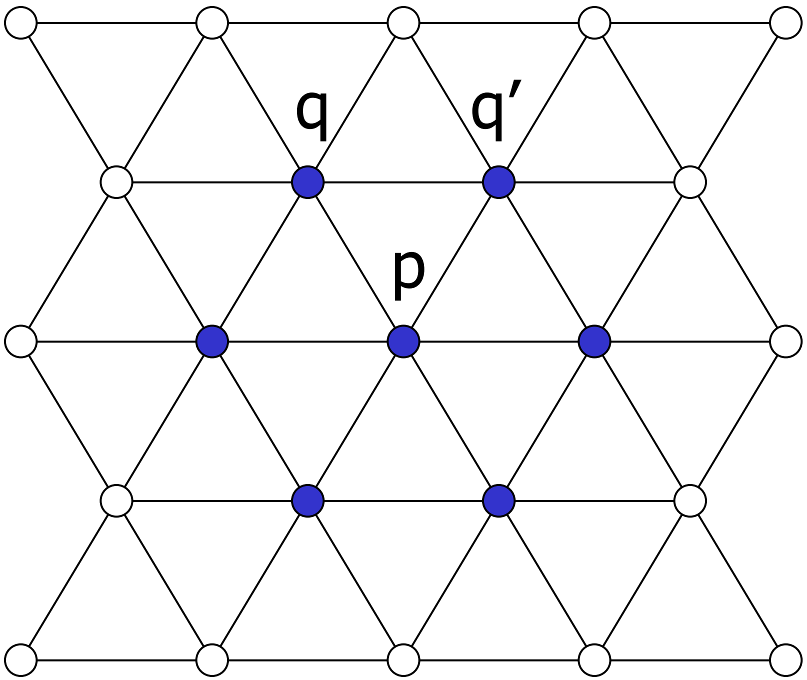
A simple example

- Focus on **2D** spin systems with $\mathbf{Z_2}$ (Ising) symmetry

- One non-trivial SPT phase, one trivial phase
  (Chen, Gu, Liu, Wen, 2011)

$\Rightarrow$ "Two kinds of Ising paramagnets"

Two kinds of Ising paramagnets



Symmetry:

$S = \prod_p \sigma^x_p$

Hamiltonians:

$H_0 = - \sum_p \sigma^x_p$

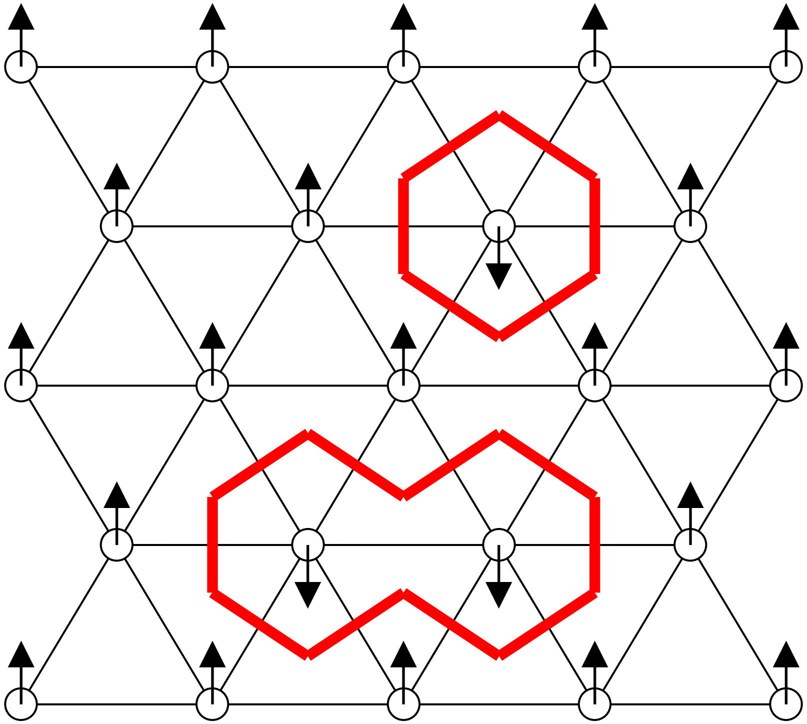
$H_1 = - \sum_p B_p \quad , \quad B_p = -\sigma^x_p \prod_{qq'} i^{(1-\sigma^z_q \sigma^z_{q'})/2}$

Two kinds of Ising paramagnets

Gd. state w.f.:

$$\Psi_0(\{\sigma^z_p\}) = 1$$

$$\Psi_1(\{\sigma^z_p\}) = (-1)^{N_{dw}}$$



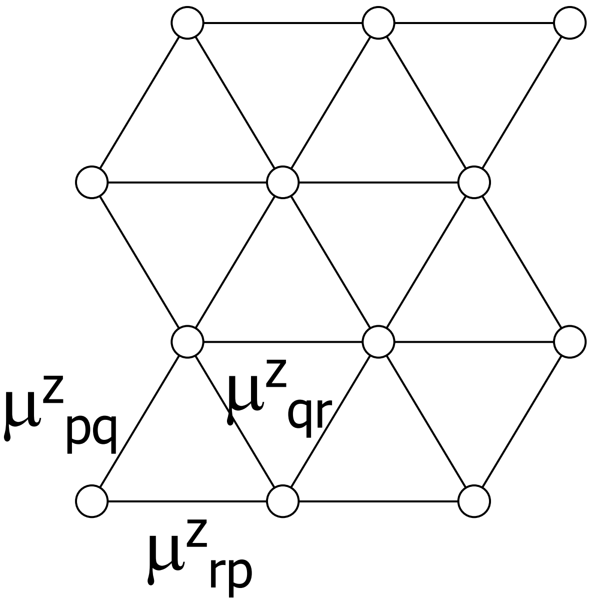
Two kinds of Ising paramagnets

1. How can we see that H_0 and H_1 belong to different phases?

2. How can we see that H_1 has a protected edge mode while H_0 does not?

Step 1: Couple to a $\mathbf{Z_2}$ gauge field

$\mathbf{Z_2}$ gauge field: $\mu^z_{pq} = \pm 1$

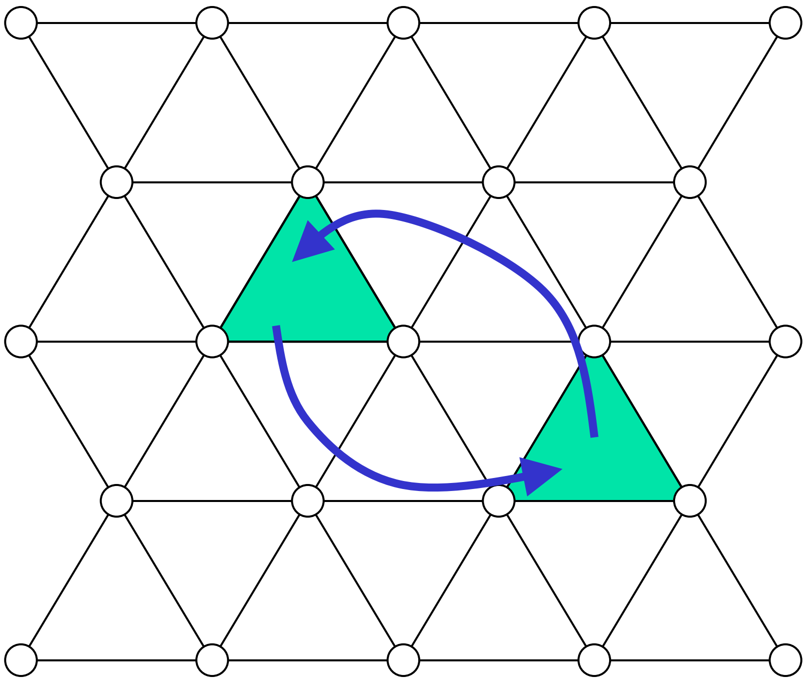


Replace: $\sigma^z_p \, \sigma^z_q \rightarrow \sigma^z_p \, \mu^z_{pq} \, \sigma^z_q$

Step 2: Compute braiding statistics of π-vortex excitations

π–vortex:

$\mu^z_{pq}\mu^z_{qr}\mu^z_{rp} = -1$



$e^{i\theta} = ?$

Result for statistics

H_0: Find $e^{i\theta} = \pm 1$

$\Rightarrow$ π-vortices are bosons or fermions

H_1: Find $e^{i\theta} = \pm i$

$\Rightarrow$ π-vortices are "semions" or "anti-semions"

Braiding statistics gives sharp distinction between H_0, H_1

Repeat program in 3D

1. Take spin/boson model with symmetry group G and "short-range entanglement"

2. Gauge the symmetry

3. Study braiding statistics of excitations in gauge theory (=Dijkgraaf-Witten theory)

4. Focus on simple case: $G = (\mathbf{Z_N})^K$

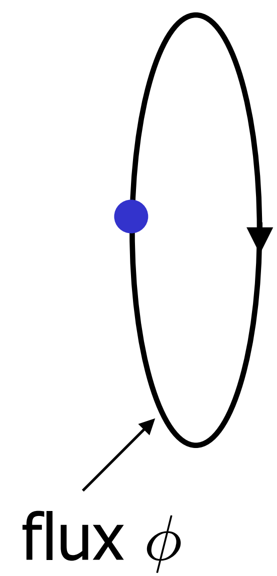
Excitations in $(\mathbf{Z_N})^K$ gauge theories

1. "Charges"

- Characterized by gauge charge:

$$q = (q_1,\ldots,q_K) \quad , \quad q_i = \text{integer (mod N)}$$

2. "Vortex loops"



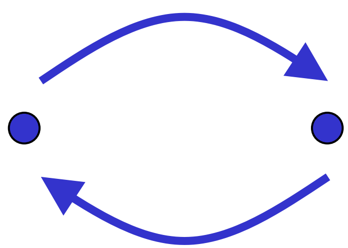
- Characterized by gauge flux:

$$\phi = (\phi_1,\ldots,\phi_K), \quad \phi_i = (2\pi/N) \cdot \text{integer}$$

- Vortex loops can also carry gauge charge

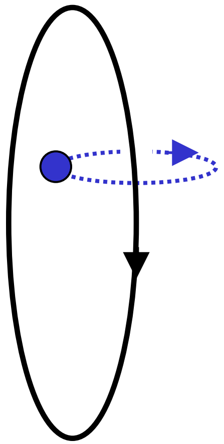
Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

Charge-charge



$\theta = 0$

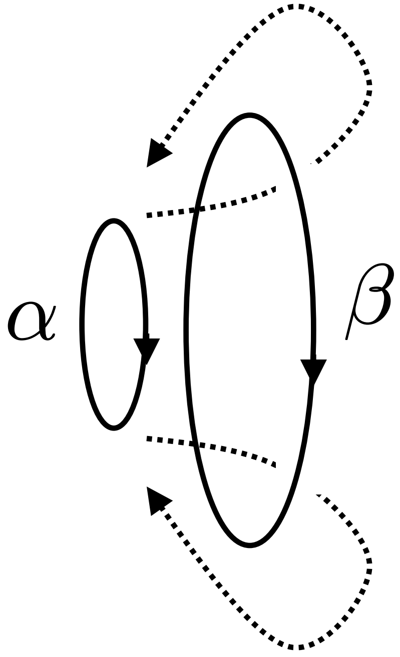
Charge-loop



$\theta = q \cdot \phi$

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

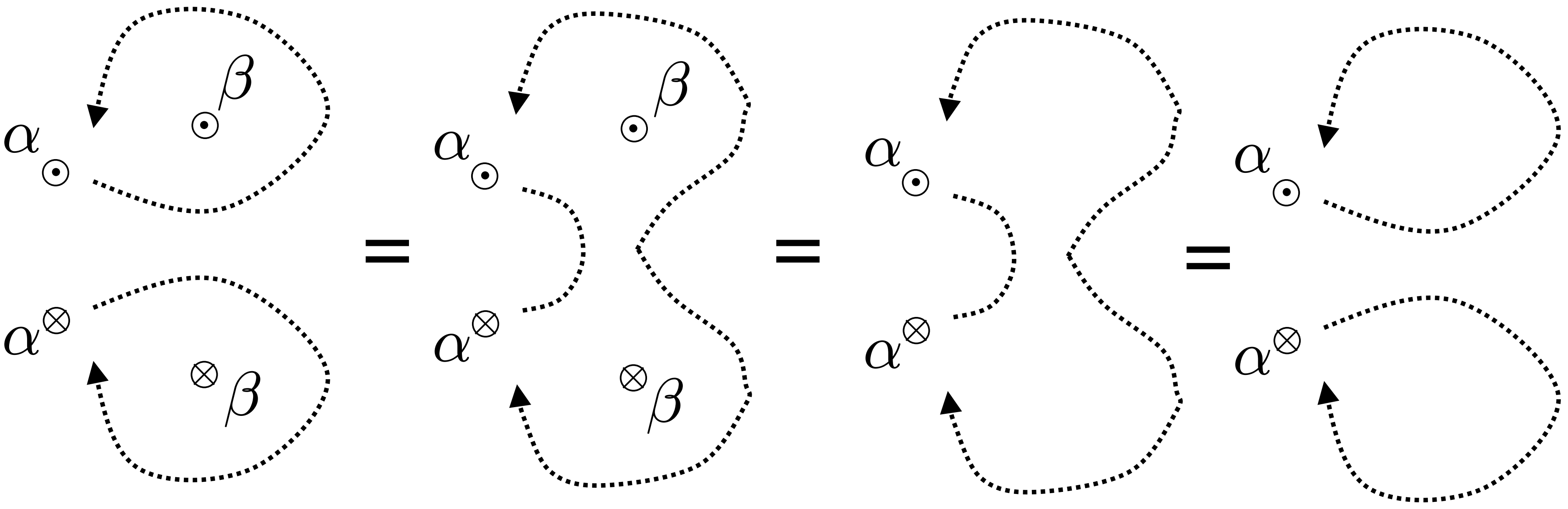
Loop-loop



$\theta_{\alpha\beta}$ = ?

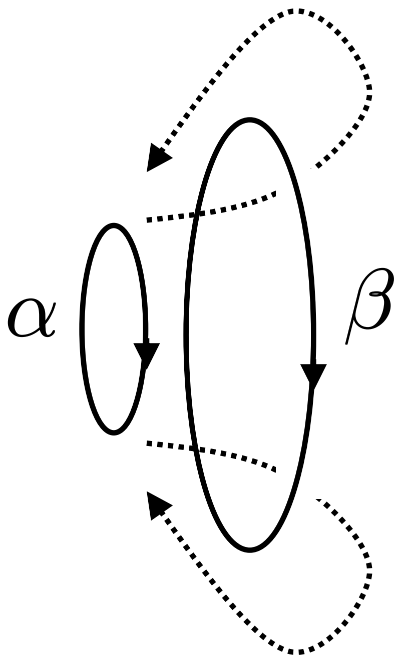
Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

If α, β are neutral:



$\Rightarrow \theta_{\alpha\beta} = 0$

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories



General case:

$$\theta_{\alpha\beta} = \mathbf{q}_\alpha \cdot \phi_\beta + \mathbf{q}_\beta \cdot \phi_\alpha$$

$\mathbf{q}_\alpha$ = charge carried by α
ϕ_α = flux carried by α

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

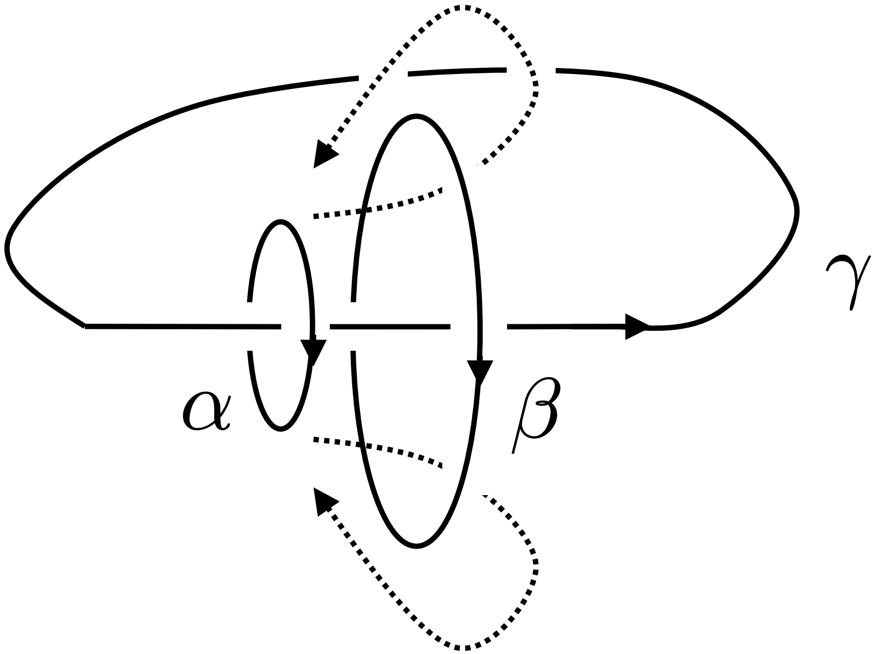
Charge-charge: $\theta = 0$

Charge-loop: $\theta = \mathrm{q} \cdot \phi$

Loop-loop: $\theta_{\alpha\beta} = \mathrm{q}_\alpha \cdot \phi_\beta + \mathrm{q}_\beta \cdot \phi_\alpha$

Independent of properties of bosonic matter!

Three-loop braiding statistics



$$\theta_{\alpha\beta,c}$$

where c = ϕ_γ

Three-loop braiding statistics

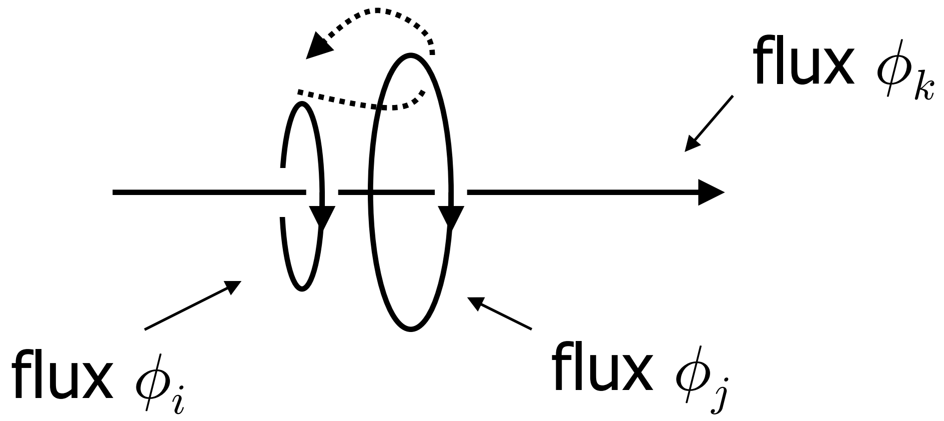
What are the physical constraints on

$\theta_{\alpha\beta,c}$?

Generalization of algebraic theory of anyons to 3D?

$$\mathrm{F} \cdot \mathrm{F} = \sum \mathrm{F} \cdot \mathrm{F} \cdot \mathrm{F}$$
$$\mathrm{R} \cdot \mathrm{F} \cdot \mathrm{R} = \mathrm{F} \cdot \mathrm{R} \cdot \mathrm{F}$$

Statistics of unit fluxes



ϕ_i = (0,…,2π/N,…,0)

Define:

$$\Theta_{ij,k} = N \cdot \theta_{\text{(above process)}} \qquad (\text{mod } 2\pi)$$

Fundamental constraints on unit flux statistics

$$\Theta_{ij,k} = \Theta_{ji,k} \qquad (1)$$

$$\Theta_{ij,k} = 2\pi/N \cdot \text{(integer)} \qquad (2)$$

$$\Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} = 0 \qquad (3)$$

Example: $\mathbf{Z_N \times Z_N}$

N^2 different exactly solvable lattice spin models labeled by (p_1, p_2). Statistics:

$$\Theta_{11,1} = 0$$
$$\Theta_{12,1} = 2\pi\, p_1/N$$
$$\Theta_{22,1} = -4\pi\, p_2/N$$
$$\Theta_{11,2} = -4\pi\, p_1/N$$
$$\Theta_{12,2} = 2\pi\, p_2/N$$
$$\Theta_{22,2} = 0$$

3-loop statistics distinguishes different SPT phases

Summary

- 2D/3D gauged SPT phases can be characterized by their braiding statistics
- 3D case requires "three-loop" statistics $\theta_{\alpha\beta,\gamma}$