Holography and the (Exact) Renormalization Group

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Introduction

- An appealing aspect of holography is its interpretation in terms of the renormalization group of quantum field theories — the ‘radial coordinate’ is a geometrization of the renormalization scale.
- Its simplest incarnation is for CFTs

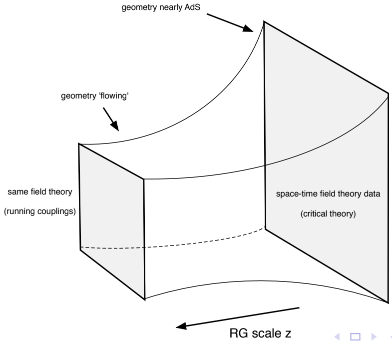
$$\begin{array}{rcl} AdS_{d+1} & \leftrightarrow & CFT_d \\ \textit{isometries} & \leftrightarrow & \textit{global symmetries} \\ \textit{scale isometry} & \leftrightarrow & \textit{RG invariance} \end{array}$$

- Usually, the correspondence is in terms of

$$\begin{array}{rcl} \textit{weakly coupled gravity} & \leftrightarrow & \textit{strongly coupled QFT} \\ \hbar & \leftrightarrow & \frac{1}{N} \sim \left(\frac{\ell_{Pl}}{\ell_{AdS}}\right)^4 \end{array}$$

Introduction

- We regard gravity as a small sector of a much bigger theory, such as a string theory (although most CMT applications ignore this...).
- More generally, RG flows (couplings and correlators changing as we coarse-grain) correspond to specific geometries that have scale isometry only asymptotically.



The Holographic Dictionary

- In a field theory, we have operators. We can talk about adding them to the action, with a corresponding coupling, and we can talk about their expectation values.
- In a CFT, operators have well-defined scaling properties

$$\hat{\mathcal{O}}_z(x) \to \lambda^{\Delta} \hat{\mathcal{O}}_{\lambda z}(\lambda x)$$

- In holography, for each such operator, there is a field propagating in the geometry (satisfies classical equation of motion).
- e.g., for a scalar field, $\Phi(z; x)$, EOM is second-order PDE, and asymptotically (i.e., near the (conformal) boundary, corresponding to near criticality)

$$\Phi(z; x) \sim z^{\Delta_-} \varphi^{(-)}(x) + z^{\Delta_+} \varphi^{(+)}(x)$$

with $\Delta_\pm$ determined by mass of field

The Holographic Dictionary

- Given

$$\Phi(z;x) \sim z^{\Delta_-}\varphi^{(-)}(x) + z^{\Delta_+}\varphi^{(+)}(x)$$

- The correspondence is:

$$\begin{aligned} \varphi^{(-)}(x) \quad &\to \textit{source} \qquad \langle ... e^{-\int_x \varphi^{(-)}(x)\hat{\mathcal{O}}(x)}\rangle \\ \varphi^{(+)}(x) \quad &\to \textit{expectation value} \qquad \langle \hat{\mathcal{O}}(x)\rangle \\ \Delta_+ \quad &\to \textit{operator scaling dimension} \end{aligned}$$

- This applies to all types of fields

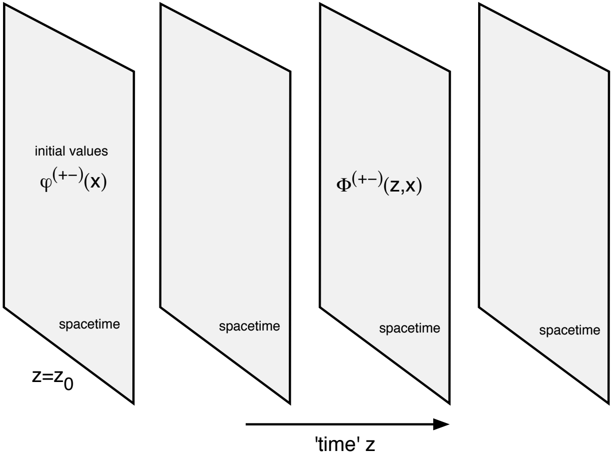
$$\begin{aligned} \textit{gauge field } A_\mu(z;x) \quad &\to \textit{conserved charge current } \hat{j}^\mu(x) \\ \textit{graviton } g_{\mu\nu}(z;x) \quad &\to \textit{conserved en} - \textit{mom tensor } \hat{T}^{\mu\nu}(x) \end{aligned}$$

- local symmetry in bulk $\to$ conserved quantity in field theory

Hamilton-Jacobi Interpretation

- Of course, second order equations can be written as a pair of first order equations
- Thus, there is a Hamiltonian formalism, but with radial coordinate z playing the role of time. (physical time remains one of the field theory coordinates)
- Source $\varphi^{(-)}(x)$ and expectation value $\varphi^{(+)}(x)$ are (boundary values of) canonically conjugate pairs.
- This fits well with Hamilton-Jacobi theory, which can be thought of as a Dirichlet problem – specify initial values — determine time-dependence of canonical variables.

Hamilton-Jacobi Interpretation



Hamilton-Jacobi Interpretation

- In this picture, the ’Hamilton equations’ ought to correspond to RG equations — how things change as we change scale, or coarse-grain.
- [de Boer, Verlinde^2 ’99]
- [Skenderis ’02, Heemskerk & Polchinski ’10, Faulkner, Liu & Rangamani ’10 ...]
- If the bulk dynamics $\leftrightarrow$ Hamilton-Jacobi, what is the ’Hamiltonian’?

$$\frac{\partial}{\partial z} S_{HJ} = -H$$

- This should encode the entire set of RG equations.
- But can this be formulated in strong coupling?

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, arXiv:1402.1430v2 [hep-th]
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]

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    have O(N)-singlet conserved currents
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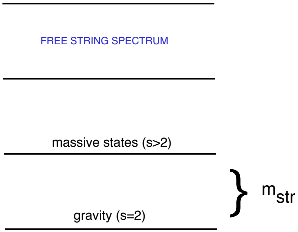
Punch Lines

- We will study free field theories perturbed by arbitrary bi-local 'single-trace' operators ($\to$ still 'free', but the partition function generates all correlation functions).
- We identify a formulation in which the operator sources correspond (amongst other things) manifestly to a connection on a *really big* principal bundle — related to 'higher spin gauge theories'
- The 'gauge group' can be understood directly in terms of field redefinitions in the path integral, and consequently there are exact Ward identities that correspond to ERG equations.
- This can be formulated conveniently in terms of a jet bundle.
- The space-time structure extends in a natural way (governed by ERG) to a geometric structure over a spacetime of one higher dimension, and *AdS* emerges as a geometry corresponding to the free fixed point.

Relation to Standard Holography?

- it's often conjectured that the higher spin theory is some sort of tensionless limit of a string theory
- not clear that this can make any sense
- however, one does expect that interactions give anomalous dimensions to almost all of the higher spin currents
- in the bulk, the higher spin symmetries are Higgsed, and the higher spin gauge fields become massive



- Dream: derive geometry of weakly coupled field theory, turn on interactions and follow to strong coupling
- Not clear what the analogue of this might be in terms of string theory (rather than higher spin theory).

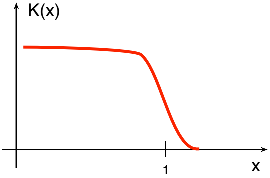
Majorana Fermions in $d = 2 + 1$

- To be specific, it turns out to be convenient to first consider the free Majorana fixed point in $2 + 1$. This can be described by the regulated action

$$S_0 = \int_x \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}\psi^m(x) = \int_{x,y} \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}(x,y)\psi^m(y)$$

- Here $P_{F;\mu}$ is a regulated derivative operator [Polchinski '84]

$$P_{F;\mu}(x,y) = K_F^{-1}(-\Box/M^2)\partial_\mu^{(x)}\delta(x,y)$$



Majorana Fermions in $d = 2 + 1$

- In 2+1, a complete basis of 'single-trace' operators consists of

$$\hat{\Pi}(x, y) = \widetilde{\psi}^m(x)\psi^m(y), \quad \hat{\Pi}^\mu(x, y) = \widetilde{\psi}^m(x)\gamma^\mu\psi^m(y)$$

- We introduce bi-local sources for these operators in the action

$$S_{int} = \frac{1}{2}\int_{x,y} \widetilde{\psi}^m(x)\Big(A(x, y) + \gamma^\mu W_\mu(x, y)\Big)\psi^m(y)$$

- One can think of these as collecting together infinite sets of local operators, obtained by expanding near $x \to y$. This quasi-local expansion can be expressed through an expansion of the sources

$$A(x, y) = \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x)\partial^{(x)}_{a_1}\cdots\partial^{(x)}_{a_s}\delta(x - y)$$

(similarly for W_μ). The coefficients are sources for higher spin local operators.

The $O(L_2)$ symmetry

- the full action takes the form

$$\begin{aligned} S &= \frac{1}{2}\int_{x,y} \widetilde{\psi}^m(x)\left[\gamma^\mu(P_{F;\mu}+W_\mu)(x,y)+A(x,y)\right]\psi^m(y) \\ &\equiv \widetilde{\psi}^m\cdot\left[\gamma^\mu(P_{F;\mu}+W_\mu)+A\right]\cdot\psi^m \end{aligned}$$

- Now we consider the following map of elementary fields

$$\psi^m(x)\mapsto\int_y \mathcal{L}(x,y)\psi^m(y)$$

- The ψ^m are just integration variables in the path integral, and so this is just a trivial change of integration variable. I'm using here the same logic that might be familiar in the Fujikawa method for the study of anomalies.
- So, we ask, what does this do to the partition function?

The $O(L_2)$ symmetry

- We look at the action

$$\begin{aligned} S &= \widetilde{\psi}^m \cdot \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \psi^m \\ &\to \widetilde{\psi}^m \cdot \mathcal{L}^T \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \mathcal{L} \cdot \psi^m \qquad (1) \\ &= \widetilde{\psi}^m \cdot \gamma^\mu \mathcal{L}^T \cdot \mathcal{L} \cdot P_{F;\mu} \cdot \psi^m \qquad (2) \\ &\quad + \widetilde{\psi}^m \cdot \left[\gamma^\mu (\mathcal{L}^T \cdot \left[P_{F;\mu}, \mathcal{L}\right] + \mathcal{L}^T \cdot W_\mu \cdot \mathcal{L}) + \mathcal{L}^T \cdot A \cdot \mathcal{L}\right] \cdot \psi^m \end{aligned}$$

- Thus, if we take $\mathcal{L}$ to be orthogonal, $\mathcal{L}^T \cdot \mathcal{L}(x,y) = \int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \delta(x,y)$, the kinetic term is invariant, while the sources transform as

$O(L_2)$ gauge symmetry

$$\begin{aligned} W_\mu &\mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot \left[P_{F;\mu}, \mathcal{L}\right] \\ A &\mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L} \end{aligned}$$

The $O(L_2)$ symmetry

- Note what is happening here: the $O(L_2)$ symmetry leaves invariant the (regulated) free fixed point action. W_μ is interpreted as a gauge field (connection) for this symmetry, while A transforms tensorily. $D_\mu = P_{F;\mu} + W_\mu$ plays the role of covariant derivative.
- More precisely, the free fixed point corresponds to any configuration

$$(A, W_\mu) = (0, W_\mu^{(0)})$$

where $W^{(0)}$ is any flat connection

$$dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0$$

- Write ERG equations – cutoff independence of $Z[M; A, W_\mu]$ leads to equations expressing scale dependence of W_μ, A
- this can be studied systematically by extending $O(L_2)$ to $CO(L_2)$, $\mathcal{L}^T \cdot \mathcal{L} = \lambda^2 1$, i.e., by including local scale transformations

The RG Equations

- These equations have the form

$$\partial_z \mathcal{A} + [\mathcal{W}_z, \mathcal{A}] = \beta^{(\mathcal{A})}$$
$$\partial_z \mathcal{W}_\mu - [P_{F;\mu}, \mathcal{W}_z] + [\mathcal{W}_z, \mathcal{W}_\mu] = \beta_\mu^{(\mathcal{W})}$$

- get 'gauge theory' in $d+1$ dimensions
- fixed point (zero of β-fns $\leftrightarrow$ flat connection)
- flat connection $\leftrightarrow$ AdS geometry
- gauge group $\leftrightarrow$ higher spin symmetry

Hamilton-Jacobi Structure

- Indeed, if we identify $Z = e^{iS_{HJ}}$, then a fundamental relation in H-J theory is

$$\frac{\partial}{\partial z} S_{HJ} = -\mathcal{H}$$

- We can thus read off the Hamiltonian of the theory, for which the RG equations are the Hamilton equations

$$\begin{aligned} \mathcal{H} &= -\mathrm{Tr}\Big\{ \Big(\Big[\mathcal{A}, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta^{(\mathcal{A})} \Big) \cdot \mathcal{P} \Big\} \\ &\quad -\mathrm{Tr}\Big\{ \Big(\Big[P_{F;\mu} + \mathcal{W}_\mu, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta_\mu^{(\mathcal{W})} \Big) \cdot \mathcal{P}^\mu \Big\} \\ &\quad -\frac{N}{2}\mathrm{Tr}\Big\{ \Big(\Delta^\mu \cdot \widehat{\mathcal{W}}_\mu + \Delta^z \cdot \widehat{\mathcal{W}}_{\underline{e}_z^{(0)}} \Big) \Big\} \end{aligned} \tag{3}$$

- Note that this is linear in momenta — the hallmark of a free theory.
- Encodes all information (concerning $O(N)$ singlet operators) in the field theory.

Remarks

- We have seen how the rich symmetry structure of the free-fixed point allows us to geometrize RG.
- The resulting structure has striking similarities with Vasiliev higher spin theory, and begs for a more precise matching.
- The β-functions encode the information about three-point functions c_{ijk}, which correspond to interactions in the bulk.
- There are many generalizations of this scheme (e.g., to fixed points with different symmetries/properties) that give rise to higher spin theories with no Vasiliev analogue. [hep-th:1404.xxxx]

Remarks

- Interactions? The partition function of the interacting fixed point in $d = 3$ can be studied by an integral transform. That is, multi-trace interactions can be induced by reversing the Hubbard - Stratanovich trick. This transform has a large-N saddle corresponding to the (fermionic) critical $O(N)$ model.
- (otherwise, N did not have to be large!)

$$Z_{crit.} = \int [dA] e^{-\int A^2} Z[M; A, W_\mu]$$

$$A(x, y) = \sigma(x)\delta(x, y)$$

- of course, another way to go after interactions is to just include sources for higher dimensional operators.
- Expect leading relevant operators will dominate, and large N factorization will lead to a fully dynamical system.

Remarks, cont.

- What of standard gravitational holography?
- The standard higher spin lore is expected to kick in here — when interactions are included, the higher spin symmetry breaks (the operators get anomalous dimensions). At strong coupling, all that is left behind is gravity.
- It is an interesting challenge to show that precisely this happens generically.