Spin MTC and Fermionic QH States

a joint-work with N. Read

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Fermionic QH States

- Laughlin ν=1/3, MR State, Read-Rezayi States (RR M=1, Milovanovic-R. SCFT)

  Bosonic versions (RR M=0) are modeled by unitary modular tensor categories

- Use spin modular tensor categories to model their topological properties

Category

A category is a directed graph such that (Vertices=objects, edges from x to y are called the morphism set Hom(x,y))

- Each vertex x has a loop id_x
- Every two compatible edges $f:x\to y, g: y\to z$ completed to a triangle with $fg:x\to z$, called composition
- Id_x is a two-sided unit for composition

(Unitary) Fusion Category

- A finite label set L, $i \in L$, dual label (anti-particle type): $i \to i^*$, and $i^{**}=i$, trivial $0=|gs>$, $0^*=0$

- Objects isomorphic to finite sum of labels
- Hom(x,y) is a f.d. Hilbert space

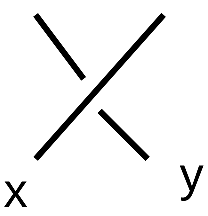
- x simple if dimHom(x,x)=1, labels are iso classes of simple objects or a set of representatives.

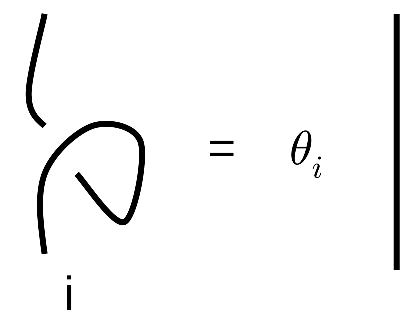
Fusion rules: $i \otimes j = \oplus N_{ij}^k\, k$ (tensor product)

Ribbon Fusion Category

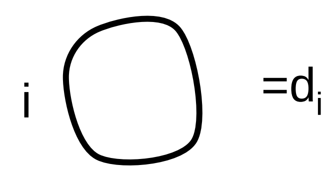
- Braiding: $c_{x,y}$: $x\otimes y \rightarrow y\otimes x$
- A twist θ_x: $x \rightarrow x$
- For a label, $\theta_i = e^{2\pi i h_i}$,

  h_i=conformal weight mod 1





Topological invariant Z(L) for labeled oriented links L, e.g.

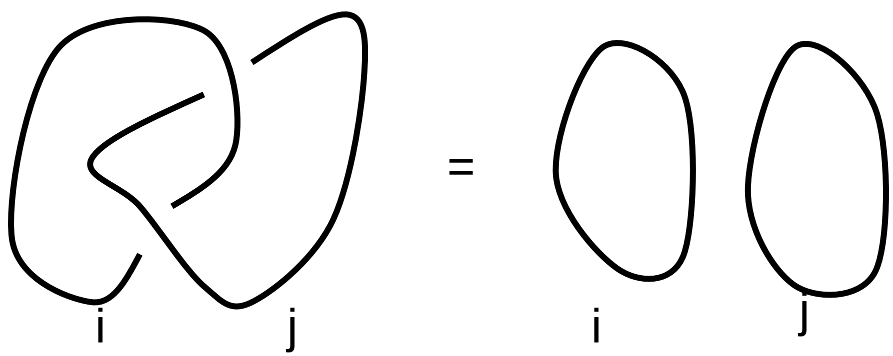


Modular Tensor Category (MTC)

A ribbon fusion category with no degenerate simple objects:

a nontrivial simple object i (=q.p. or q.h. mod local) is degenerate or transparent if $Ds_{ij}=d_i d_j$ for any other simple j

Hopf Link



$D^2 = \sum d_i^2$

D total quantum dim

D^2 total order or index?

Ribbon Fusion Category

If not modular, braid statistics and invariants of closed 3-manifolds are well-defined,

but the full TQFT structure such as the rep of the modular group "SL(2,Z)" cannot be defined.

If modular, then the modular s-matrix $s=(s_{ij})$ is not singular, and the full (2+1)-TQFT is defined.

E.g. Fusion rules are given by Verlinde formulas.

Spin MTC

- An MTC with a fermion f
- A fermion is a simple object $f^2=1$ and $\theta_f=-1$
- Note if $f^2=1$, then $\theta_f=1$ (boson), $\theta_f=-1$ (fermion), or $\theta_f=\pm$ i (semion)
- In Ising MTC with simples $1,\sigma,\psi$,

  ψ is a fermion.

Examples

- Laughlin states $\nu=1/Q$,

  Z_{4Q}, $L=\{0,1,\ldots,4Q-1\}$, $Q=$odd, $f=2Q$,

  charge $q_s=s/2Q$, $\theta_s=e^{2\pi i s^2/8Q}$
- MR state

  Ising $\times$ Z_8, Z_8=Laughlin at $Q=2$

  $L=\{x\otimes s\}$, $x=1,\sigma,\psi$, $s=0,1,\ldots,7$

  $f=\psi\otimes 4$, charge from Z_8
- $SU(2)_k$, $k=2 \bmod 4$, e.g. $k=6$, $L=\{0,1,\ldots,6\}$, $f=6$

c-Spin MC

- A c-spin MC is a spin MTC (F, f) which is unitary and covers a fermionic QH state (tentative).

- c-Spin MC

  $\downarrow$ Local w.r.t fermion---NS vs R

  Unitary ribbon fusion category

  $\downarrow$ Identify fermion with |gs>

  Tensor category

  (without braidings and twists,

  but with pure braidings and double twists)

f is a charged fermion

Covering theory (F,f) has two components:

- Statistics/Spin sector: B (UMTC)

  Charge sector: C (cyclic UMTC)

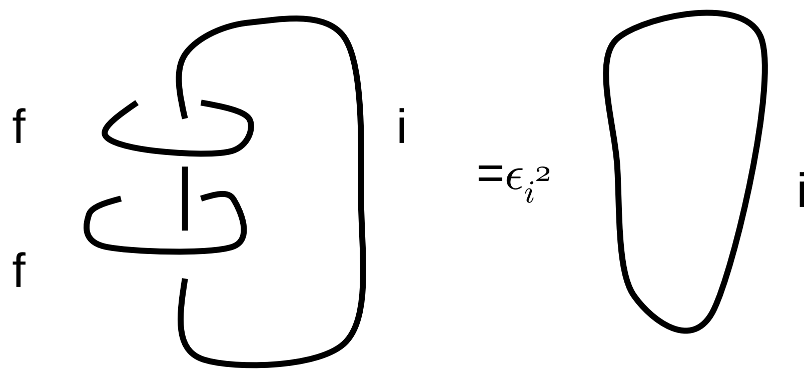
  Topological content is in the quotient Q of $B \otimes C$, where every anyon has a well-defined electric charge mod 1.
- More general, F is not a product.

Z_2 Grading

- Unitary spin MTC, for any $i \in L$,

  $Ds_{fi}=\varepsilon_i \, d_i$, $\varepsilon_i=\pm 1$

  i is even if $\varepsilon_i=1$, odd otherwise
- $L=L_0 \cup L_1$, $f \in L_0$
- If $N_{ij}^{k} \neq 0$, then $\varepsilon_i \varepsilon_j \varepsilon_k=1$



Elementary Properties

- $D_0^2 = D_1^2$, where $D_k^2 = \sum_{i \in L_k} d_i^2$
- $s_{fi,j} = \epsilon_j\, s_{i,j}$
- $\theta_{fi} = -\epsilon_i\, \theta_i$
- f has no fixed points L_0
- Proofs:

1st and fth row are orthogonal, so $\sum s_{f,i} d_i = 0$, i.e., $\sum_i \varepsilon_i\, d_i^2 = 0$

Verlinde s-matrix

Define naive fusion rules

$N_{[a][b]}^{[c]}=N_{ab}^{c}+N_{ab}^{fc}$,

where [a] are quotient labels,

and naive s-matrix,

$s^{Q}_{[a],[b]}=2s_{a,b}$, then

s^Q is a unitary matrix and Verlinde formulas hold for the naive fusion rules.

MR Fusion Rules

Labels L_Q={1, ψ, σ, σ', α, α'}
charges={0, 0,1/4,3/4, 1/2,1/2}

$\alpha\alpha'=1$, $\sigma\sigma'=1+\psi$

$\psi^2=1$, $\alpha^2=\alpha'^2=\psi$, $\sigma^2=\sigma'^2=\alpha+\alpha'$,

$\psi\sigma=\sigma$, $\psi\sigma'=\sigma'$, $\psi\alpha=\alpha'$, $\alpha\sigma=\sigma'$

No braided fusion category realizations!

(P. Bonderson's thesis)

MR s^Q-matrix

$D=2\sqrt{2}$

$s^Q=1/D$

$$\begin{pmatrix} 1 & \sqrt{2} & 1 & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & i\sqrt{2} & 0 & -i\sqrt{2} \\ 1 & -\sqrt{2} & 1 & 1 & -\sqrt{2} & 1 \\ 1 & i\sqrt{2} & 1 & -1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & -\sqrt{2} & -i\sqrt{2} & 0 & i\sqrt{2} \\ 1 & -i\sqrt{2} & 1 & -1 & i\sqrt{2} & -1 \end{pmatrix}$$

A Puzzle

$SU(2)_6$, labels $L=\{0,1,2,3,4,5,6\}$, $f=6$,
$L_0=\{0,2,4,6\}$, $L_1=\{1,3,5\}$

$$s^Q=1/D \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}$$

$[0]=1$, $[2]=x$, then $x^2=1+2x$

Verlinde formulas hold for the fusion rules,
but $\{1,x\}$ with $x^2=1+2x$ does not exist as a fusion category (V. Ostrik)

(2+1)-TQFTs from MTCs

Two compatible functors (rules)

- A modular functor V:

  surfaces Y to Hilbert spaces V(Y),

  mapping classes b: $Y \to Y$ to unitary maps

  $V(b): V(Y) \to V(Y)$

- A partition functor Z:

  bordisms M^3 from Y_1 to Y_2 to

  linear maps $Z(M^3): V(Y_1) \to V(Y_2)$

$Y_i = \emptyset$, $Z(M^3)$=partition functions in CSW theories

Spin TQFTs from spin MTCs

- Surfaces and 3-mfds are endowed with compatible spin structures
- Spin structure:

  given an oriented surface Y, a spin structure σ on Y is a quadratic enhancement $q_\sigma: H_1(Y,Z_2) \rightarrow Z_2$ such that $q(x+y)=q(x)+q(y)+\langle x,y\rangle \bmod 2$, where $\langle x,y\rangle$ is the Z_2-intersection form of Y.

Theorem (C. Blanchet)

Given a TQFT, a spin structure σ on closed oriented surface Y,

let $V^s(Y,\sigma)=\{v\in V(Y) \mid O_\gamma v=(-1)^{q_\sigma(\gamma)}v, \text{ all } \gamma\}$, where γ is a simple closed curve γ on Y and O_γ is an operator,

Then $V(Y)=\sum_{\text{spin structures } \sigma} V^s(Y,\sigma)$

Quotient Categories

- Quotient $F \rightarrow Q$

  Let $\Gamma = 1 \oplus f$,

  Objects of Q=objects of F,

  Given objects x, y of Q,
  $Hom_Q(x,y) = Hom_F(\Gamma \otimes x, y)$
- Note that in Q, $f \cong 1$
- $\otimes$ is well-defined
- ? direct sum, semi-simplicity, rigidity

Other Structures

- Braiding, No
- Twist, No
- Pure braidings, Yes
- Double twists, Yes
- Representation of the subgroup of SL(2,Z) generated by s and t^2

Possible Applications

- Entanglement entropy: $-\log D_Q$

- Topological stability