Periodic table for topological insulators and superconductors

Main themes:

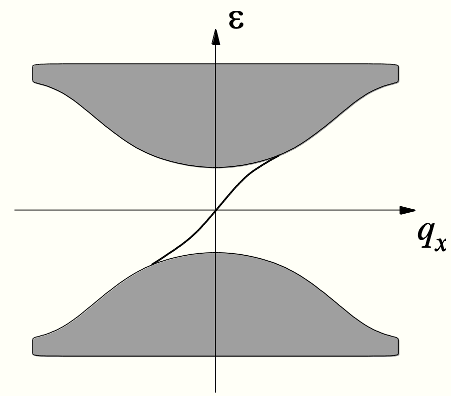
Periodic table:

ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

			Ti=50	Zr=90	?=180.
			V=51	Nb=94	Ta=182.
			Cr=52	Mo=96	W=186.
			Mn=55	Rh=104,4	Pt=197,
			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pl=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be=9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,4	?=68	Ur=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,5	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

Topological features of electron spectrum



Bott periodicity:

$$\widetilde{K}^{n+2}(X) \cong \widetilde{K}^{n}(X)$$

$$\widetilde{KO}^{n+8}(X) \cong \widetilde{KO}^{n}(X)$$

The elements: gapped free-fermion phases

- IQHE systems (GaAs-AlGaAs heterostructures, graphene)
- 2D spin-Hall insulators (HgTe)
- 3D topological insulators (BiSb)
- 1D triplet superconductors ($(\mathrm{TMTSF})_2\mathrm{X}$)
- $p_x + ip_y$ superconductors (SrRu)
- Superfluid ^{3}He-B
- 0D systems; topologically trivial phases in all dimensions

Yet undiscovered:

- $(p_x + ip_y)\uparrow + (p_x - ip_y)\downarrow$ superconductors
- Majorana chains

The table(s)

The first table: Complex K-theory (described by C_0, C_1)

Stable equivalence classes of complex vector bundles on a topological space X are given by homotopy classes of maps $X \to C_0$.

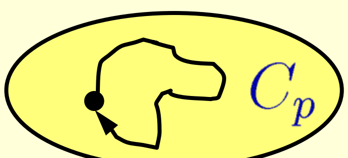
p	Classifying space C_p	$\pi_0(C_p)$	invariant
0	$\left(\lim\limits_{n,m\to\infty} U(n+m)/(U(n)\times U(m))\right) \times \mathbb{Z}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
1	$\lim\limits_{n\to\infty} U(n)$	0	

C_0 parametrizes linear subspaces (or rather, *differences* between such subspaces). The $\mathbb{Z}$ factor keeps track of the difference in dinesion.

C_1 parametrizes unitary matrices

Bott periodicity: $\quad C_1 \sim \Omega(C_0), \quad C_0 \sim \Omega(C_1).$

loop spaces



Second table: real K-theory (described by $R_0, \ldots, R_7$)

p	Classifying space R_p	$\pi_0(R_p)$	invariant
0	$\big(O(n+m)/(O(n)\times O(m))\big)\times\mathbb{Z}$	$\mathbb{Z}$	$k\in\mathbb{Z}$
1	$O(n)$	$\mathbb{Z}_2$	$\det S=\pm 1$
2	$O(2n)/U(n)$	$\mathbb{Z}_2$	$\mathrm{Pf}\, A=\pm 1$
3	$U(2n)/Sp(n)$	0	
4	$\big(Sp(n+m)/(Sp(n)\times Sp(m))\big)\times\mathbb{Z}$	$\mathbb{Z}$	$k\in\mathbb{Z}$
5	$Sp(n)$	0	
6	$Sp(n)/U(n)$	0	
7	$U(n)/O(n)$	0	

The real Grassmanian R_0 classifies real vector bundles.

$R_p \sim \Omega^p(R_0)$; Bott periodicity: $R_{p+8}\sim R_p$.

The space R_p classifies *Clifford extensions* $\mathrm{Cl}^{p,0}\setminus\mathrm{Cl}^{p+1,0}$, which are closely related to free fermionic Hamiltonians with symmetries.

The classification

for all combinations of charge conservation (Q) and time-reveral symmetry (T)

p	$\pi_0(C_p)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		Q: IQHE	
1	0			

p	$\pi_0(R_p)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$	"wrong T": $T^2=1$	No symmetry: p_x+ip_y (SrRu)	T (^{3}He-B)
1	$\mathbb{Z}_2$	No symmetry (Majorana chain)	T-invariant triplet superconductors ($(p_x+ip_y)\uparrow+(p_x-ip_y)\downarrow$)	T and Q (BiSb)
2	$\mathbb{Z}_2$	T-inv. triplet superconductors ($(TMTSF)_2X$)	T and Q: T-invariant insulators (HgTe)	
3	0			
4	$\mathbb{Z}$			

Summary of the classification

- All phases (except "wrong T") belong to 4 symmetry classes:
  - Q (Hall insulators) — (mod 2) table
  - no symmetry (superconductors)
  - T (TRI superconductors)
  - T, Q (TRI insulators)

  $\left.\begin{array}{ll} \text{– no symmetry} & \text{(superconductors)} \\ \text{– } T & \text{(TRI superconductors)} \\ \text{– } T, Q & \text{(TRI insulators)} \end{array}\right\}$ (mod 8) table

Previous classification schemes:

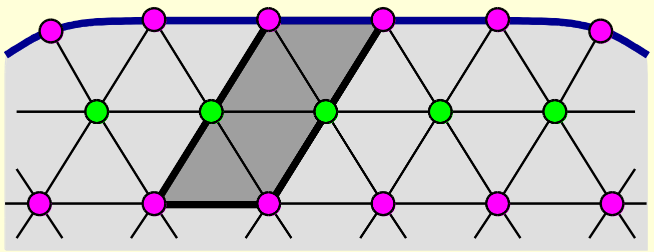
- Random matrix ensembles: the same 10 classes, but no dimension shift or periodicity.
- Recent results:
  - Qi, Hughes, Zhang (2008) $\mathbb{Z}_2$ insulators;
  - Qi, Hughes, Raghu, Zhang (2008) $\mathbb{Z}_2$ superconductors: explicit dimension shift but no grand unification.

Well-studied example: IQHE

Electromagnetic response: $j_\alpha = \sigma_{\alpha\beta} E_\beta, \quad \sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu, \quad \sigma_{xx} = 0.$

$$S = \frac{\nu}{4\pi}\int \epsilon^{\mu\nu\lambda} A_\mu \,\partial_\nu A_\lambda \, d^3x$$

Tight-binding model:



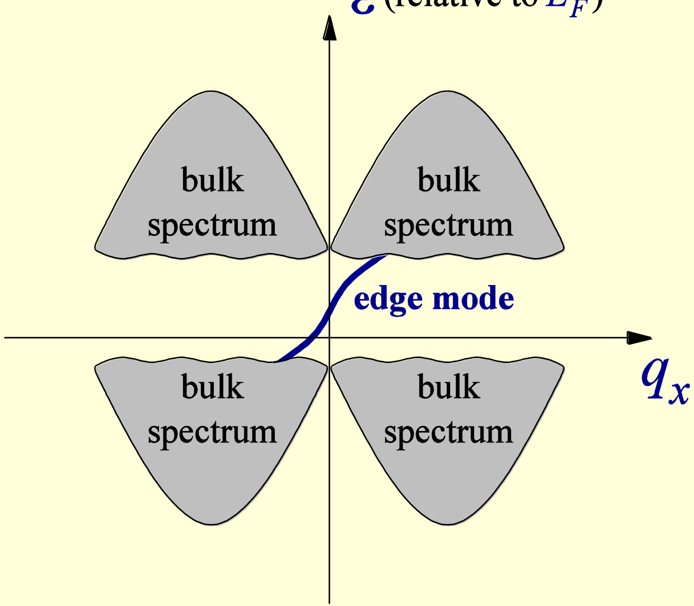
$\pi/2$ flux per triangle

unit cell $= 4$ triangles

$$\widehat{H} = \sum_{j,k} H_{j,k}\, a_j^\dagger a_k$$

$$= \sum_{j,\alpha;\,k,\beta} H_{j,\alpha;\,k,\beta}\, a_{j,\alpha}^\dagger a_{k,\beta}$$

$$= \sum_{q;\,\alpha,\beta} H_{\alpha,\beta}(q)\, a_{q,\alpha}^\dagger a_{q,\beta}$$



Spectrum: $\varepsilon(q) = \text{eigenvalue}(\ \underbrace{H(q)}_{2\times 2 \text{ matrix}}\ )$

Topological invariant: Chern number

- Abstract notion:

Occupied states: $\mathcal{L}(q) \subseteq \mathbb{C}^2$ (eigenvectors corresponding to negative eigenvalues of $H(q)$)

$$q \mapsto \mathcal{L}(q)$$

momentum space $\rightarrow$ complex Grassmanian C_0

- Calculation using spectrum flattening, or projection

$\widetilde{H}(q) = \operatorname{sgn}(H(q))$ (same eigenvectors; eigenvalues $= \pm 1$)

Note that $\widetilde{H}(q) = 1 - 2P(q)$, where $P(q)$ is the projector onto the occupied electron states.

$$\nu = \frac{1}{2\pi i} \int \operatorname{Tr}\left(P \left(\frac{\partial P}{\partial q_x} \frac{\partial P}{\partial q_y} - \frac{\partial P}{\partial q_y} \frac{\partial P}{\partial q_x} \right) \right) dq_x \, dq_y .$$

(Thouless, Komoto, Nightingale, den Nijs, 1982)

Dealing with disorder: Real-space expression for the Chern number

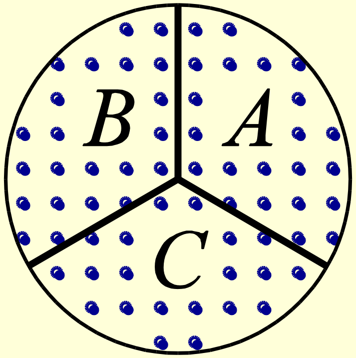
- If electrons are localized, the Hall conductivity is quantized (rigorous proof by Bellissard at al (1994))

- Let $\widehat{H} = \sum_{j,k} H_{jk} a_j^\dagger a_k$. A sufficient localization condition is this:

spatial dimension

$$\widetilde{H}_{jk} \le c\left|\vec{r}_j - \vec{r}_k\right|^{-(2+\varepsilon)}, \quad \text{where } \widetilde{H} = \operatorname{sgn} H. \qquad (\widetilde{H}^2 = 1,\ \ \widetilde{H}^\dagger = \widetilde{H})$$

- A quantized quantity (related to the Hall conductivity in the adiabatic approximation) may be defined as follows:

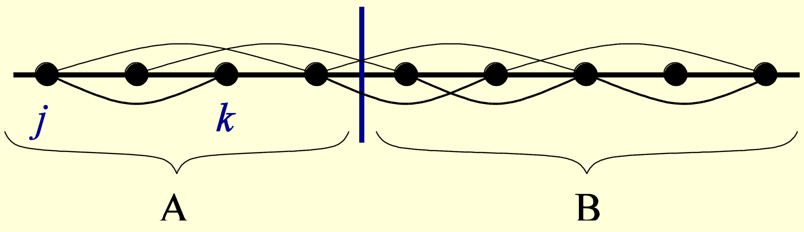


$$\nu = \sum_{j\in A} \sum_{k\in B} \sum_{l\in C} h_{jkl}, \qquad \text{where}$$

$$h_{jkl} = 12\pi i\left(P_{jk}P_{kl}P_{lj} - P_{jl}P_{lk}P_{kj}\right) \qquad \left(P = \frac{1}{2}(1 - \widetilde{H})\right)$$

One-dimensional analogue:

Matrix U on an infinite chain:



- U is unitary;
- U is *local*:
  $|U_{jk}| \leq c|j-k|^{-(1+\varepsilon)}$.

Definition: $f_{jk} = |U_{jk}|^2 - |U_{kj}|^2$ — "current" from j to k.

Theorem: The current is conserved: $\sum_k f_{jk} = 0$.

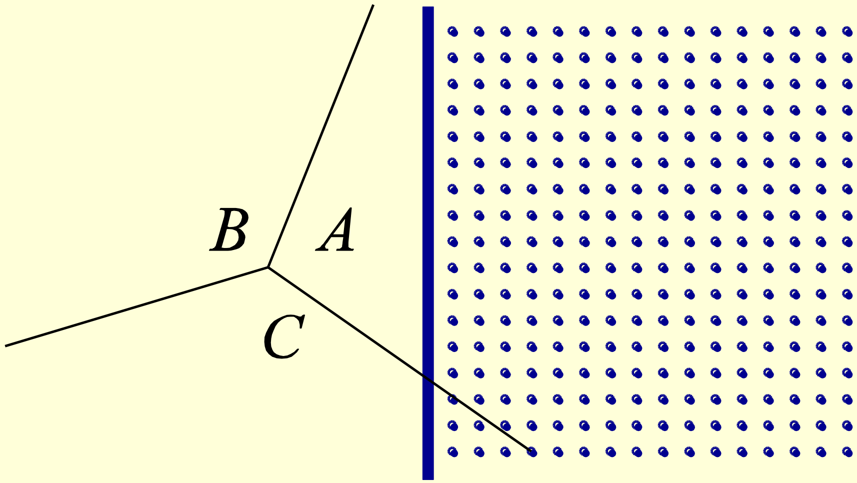
The total flow, $\mathcal{F} = \sum_{j \in A} \sum_{k \in B} f_{jk}$
is *constant* along the chain and has *integer* value.

Corollary: It is impossible to define a local unitary matrix on a *half-infitite* chain.

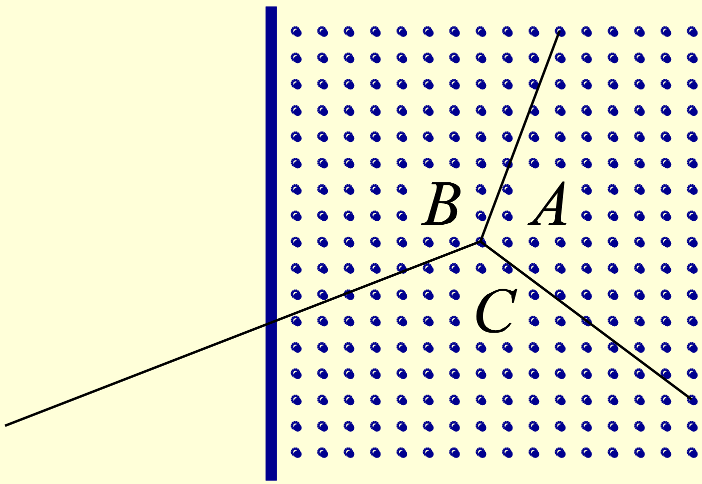
(The faliure of unitarity at the end of the chain is described by the difference of two projectors.)

Application to IQHE

- Similarly, the real-space Chern number is constant if the electrons are localized everywhere. Thus they must be **delocalized** at the edge.



$\nu = 0$



$\nu \neq 0$

- This is an abstract proof: It does not really tell what happens at the edge.

Classification problem

This is a mathematical problem. We need to define:

- Admissibe phase: gapped free fermions in $\mathbb{R}^d$. (The energy gap is not essential; a localization condition will suffice. We may also consider systems on a compact space X, which is big compared to the localization length.)

- Equivalence between admissible phases: two phases are equivalent if they can be continuosly deformed one to the other *in the presence of trivial degrees of freedom*: $H_1 \oplus T \sim H_2 \oplus T$.

- The object of classification: the *difference* between equivalence classes of admssible systems. It is an element of some Abelian group.

– The difference class should be computable, but explicit formulas are not necessary.

Three different settings

1. Continuous systems (Dirac operators).

   This is the simplest approach, and the answer is more general than one might expect. One can even describe long-range disorder as *texture*, i.e., slowly varying *mass term*.

2. Translationally invariant lattice systems (K-theory: vector bundles on $\overline{\mathbb{T}}^d$).

   Most practical, but least intuitive method. Carries some extra invariants (e.g., the Hall conductivity per layer in a 3D system). If $\overline{\mathbb{T}}^d$ is replaced by $\overline{S}^d$, then the answer is exactly the same as for Dirac operators.

3. Disordered systems (K-homology).

   Theorem. *Any localizing free-fermion Hamiltonian on a spin manifold is topologically equivalent to a texture.* (Explicit construction; need $\sim (l/a)^d$ Dirac modes.)

Free-fermion Hamiltonian

Using creation and annihilation operators:

$$\widehat{H} = \sum_{j,k} H_{jk} a_j^\dagger a_k \qquad \text{(suitable if charge is conserved)}$$

- Energy gap: $\Delta \leq |\varepsilon| \leq W$;
- Locality (for $d > 0$): The matrix elements H_{jk} vanish if $|\vec{r}_j - \vec{r}_k| > r$ (or, at least, they decay fast enough).

– The ground state is characterized by a k-dimensional subspace $\mathcal{L} \subseteq \mathbb{C}^{k+m}$ (negative-energy states). One may also use the "spectrum-flattened" matrix $\widetilde{H} = \mathrm{sgn}(H)$, which is slightly less local than H.

– Parametrized by $U(k+m)/(U(k) \times U(m))$.

Complex K-theory

- Problem: Describe *families* of Hamiltonians (in $d=0$):

$$H(q) : q \in Y, \qquad \text{where } Y \text{ is some } \textit{parameter space}.$$

- Answer: $K^0(Y) =$ homotopy classes of maps $Y \to C_0$,

$$C_0 = \bigoplus_k \lim_{n,m\to\infty} U(k+n+m)/(U(k+n)\times U(m))$$

# of negative energy states

extra dimensions added to assist the homotopy

- Another problem: Describe families of unitary matrices U

- $K^{-1}(X) = K^1(X) =$ homotopy classes of maps $Y \to C_1$,

$$C_0 = \lim_{n\to\infty} U(n)$$

- $K^0(\text{point}) = \pi_0(C_0) = \mathbb{Z}, \qquad K^1(\text{point}) = \pi_0(C_1) = 0.$

The first table --- now understood

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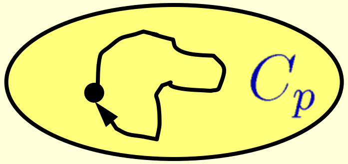


Hamiltonian using Majorana operators

$$c_{2l-1} = a_l + a_l^\dagger, \quad c_{2l} = -i(a_l - a_l^\dagger) \qquad (c_j c_k + c_k c_j = 2\delta_{jk})$$

$$\widehat{H} = \frac{i}{4}\sum_{j,k} A_{jk} c_j c_k \qquad (A \text{ is real skew-symmetric matrix})$$

– Convenient if the charge is not conserved, or in the presence of time-reversal symmetry.

– The ground state is determined by $\widetilde{A} = -i\,\mathrm{sgn}(iA)$.

– Parametrized by $O(2n)/U(n)$:

$$\widetilde{A} = S \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} S^{-1}, \qquad S \in O(2n)/U(n)$$

Symmetries

(sufficient for classification in dimension $d = 0$)

Charge operator: $\widehat{Q} = \sum_l (a_l^\dagger a_l - \frac{1}{2}) = \frac{i}{2} \sum_l c_l' c_l'' = \frac{i}{4} \sum_{j,k} Q_{jk} c_j c_k$

$$\boxed{\begin{aligned} c_l' &= a_l + a_l^\dagger \\ c_l'' &= \tfrac{1}{i}(a_l - a_l^\dagger) \end{aligned}}$$

$$Q = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Time-reversal operator (acts by conjugation: $T(x) = \widehat{T} X \widehat{T}^{-1}$)

$$\begin{aligned} T(i) &= -i \\ T(a_\uparrow) &= a_\downarrow \\ T(a_\downarrow) &= -a_\uparrow \\ T(a_\uparrow^\dagger) &= a_\downarrow^\dagger \\ T(a_\downarrow^\dagger) &= -a_\uparrow^\dagger \end{aligned}$$

$$T(c_k) = \sum_j T_{jk} c_j$$

real skew-symmetric matrix

Properties of the matrices Q and T

- Q, T are real skew-symmetric matrices
- $Q^2 = T^2 = -1, \quad QT = -TQ$
- Let $\widehat{H} = \frac{i}{4}\sum_{j.k} A_{jk} c_j c_k$.
  - If $\widehat{H}$ is Q-invariant, then $QA = AQ$;
  - If $\widehat{H}$ is T-invariant, then $TA = -AT$.

Q and T generate a *Clifford algebra*

- Let $e_1 = T, \quad e_2 = QT$, and let $\widetilde{A} = -i\,\mathrm{sgn}(iA)$.

No symmetry:	$\widetilde{A}^2 = -1$
T only:	$e_1^2 = \widetilde{A}^2 = -1, \quad e_1\widetilde{A} = -\widetilde{A}e_1$
T and Q:	$e_1^2 = e_2^2 = \widetilde{A}^2 = -1$, $e_1 e_2 = -e_2 e_1$, $e_j\widetilde{A} = -\widetilde{A}e_j \ (j = 1, 2)$

e_1, e_2 generate a Clifford algebra, and $\widetilde{A}$ *extends* it.

Extension type $\mathrm{Cliff}^{0,q} \setminus \mathrm{Cliff}^{0,q+1}$ $(q = 0, 1, 2)$

Clifford algebras (with real coefficients)

$\underline{\mathrm{Ciff}^{p,q}}$: Generators: $e_1, \ldots, e_{p+q}$

$$e_1^2 = \cdots = e_p^2 = 1, \quad e_{p+1}^2 = \cdots = e_{p+q}^2 = -1,$$
$$e_j e_k = -e_k e_j \ \ (\text{for } j \neq k)$$

Examples:

- $\mathrm{Ciff}^{1,0} \cong \mathbb{R} \oplus \mathbb{R} \quad = \left\{ x\frac{1+e_1}{2} + y\frac{1-e_2}{2} \; : \; x, y \in \mathbb{R} \right\}$
- $\mathrm{Ciff}^{1,0} \cong \mathbb{C} \quad (e_1^2 = -1 \; ; \; e_1 \text{ is like } i)$
- $\mathrm{Ciff}^{2,0} \cong \mathbb{R}(2)$ (the algebra of 2×2 real matrices, $e_1 = \sigma^z, \; e_2 = \sigma^x$)
- $\mathrm{Ciff}^{1,1} \cong \mathbb{R}(2) \quad (e_1 = \sigma^z, \; e_2 = i\sigma^y)$
- $\mathrm{Ciff}^{0,2} \cong \mathbb{H}$ (the algebra of quaternions)

Periodicity:

- $\mathrm{Ciff}^{p+1,q+1} \cong \mathrm{Ciff}^{p,q} \otimes \mathbb{R}(2) \sim \mathrm{Ciff}^{p,q}$

Morita equivalence

- $\mathrm{Ciff}^{p+8,q} \cong \mathrm{Ciff}^{p,q} \otimes \mathbb{R}(16) \sim \mathrm{Ciff}^{p,q}$

Classification by symmetry (in dimension 0)

- Technicality: Replacing the negative Clifford generators T, TQ, $\widetilde{A}$ by positive ones: $\underline{\mathrm{Ciff}^{0,q}} \sim \mathrm{Ciff}^{0,q} \otimes \mathbb{R}(2) \cong \underline{\mathrm{Ciff}^{q+2,0}}$

Morita equivalence

- Solving the extension problem for $p = q + 2$:
  Given a representation of $\mathrm{Ciff}^{p,0}$, *find all actions of an additional positive generator* e_{p+1}:

$p = q + 2$	q	Symmetries	Parameter space R_p
1	-1	"Wrong T" ($T^2 = 1$)	$O(n)$
2	0	None	$O(2n)/U(n)$
3	1	Standard T ($T^2 = -1$)	$U(2n)/Sp(n)$
4	2	T and Q	$Sp(k+m)/(Sp(k) \times Sp(m))$

First step to higher dimensions: Majorana chain

(No symmetry at all: spin-polarized superconductor)

$$\widehat{H} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$$

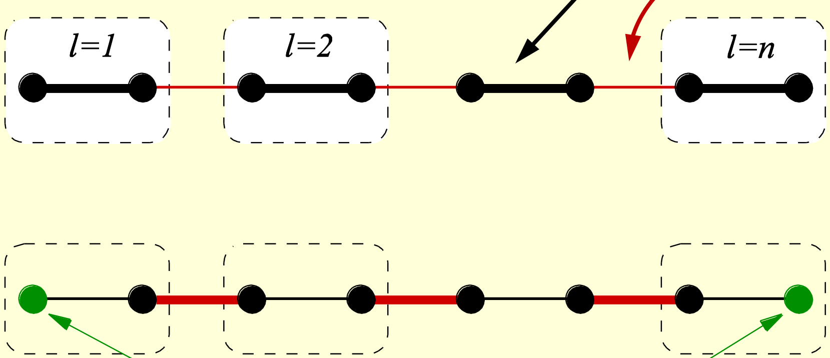
real skew-symmetric matrix

Majorana operators:

$$c_{2l-1} = a_l + a_l^\dagger, \quad c_{2l} = \tfrac{1}{i}(a_l - a_l^\dagger)$$

$$c_j c_k + c_k c_j = 2\delta_{jk}$$

For example, $H = \frac{i}{2}\left(u \sum_{l=1}^{n} c_{2l-1} c_{2l} + v \sum_{l=1}^{n-1} c_{2l} c_{2l+1} \right)$

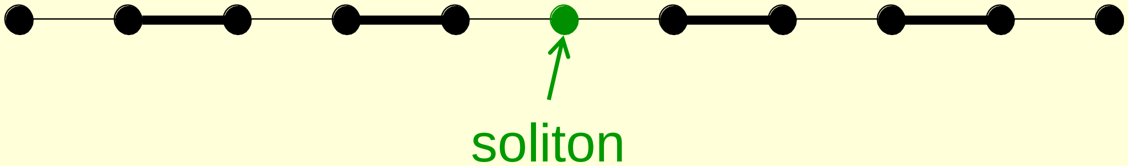


$u > v$

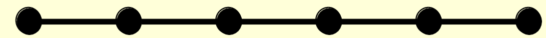
$u < v$: The ground state is 2-fold degenerate: $\Delta E \sim e^{-n/\xi}$.

upaired Majorana modes

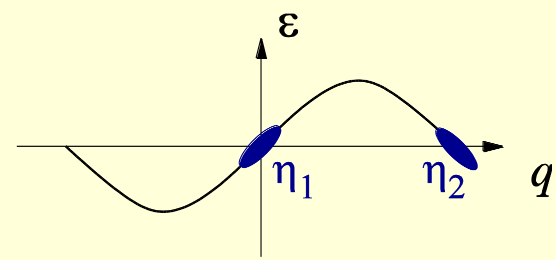
Soliton and a Dirac operator



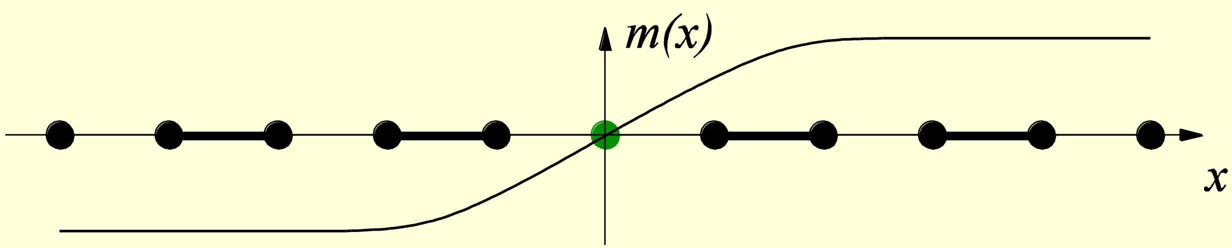
- Consider the case $u = v$ (gapless system)



- Take the continuum limit (fields η_1, η_2)



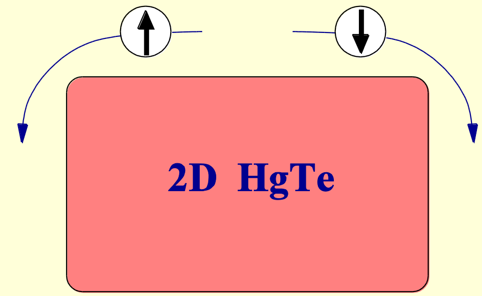
- Introduce a mass term, $m(x) \sim u - v$



$$\widehat{H} = \frac{i}{2} \int \eta^T \begin{pmatrix} \partial & m \\ -m & -\partial \end{pmatrix} \eta \, dx, \qquad \eta = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix}$$

$m(x)$ changes sign $\Rightarrow$ **zero mode** (by the index theorem)

A more realistic example



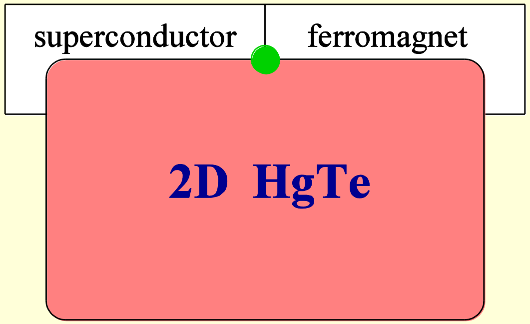
$$\widehat{H} = \int \Big(i(\psi_\uparrow^\dagger \partial \psi_\uparrow - \psi_\downarrow^\dagger \partial \psi_\downarrow)$$
$$+ h_x(\psi_\uparrow^\dagger \psi_\downarrow + \psi_\downarrow^\dagger \psi_\uparrow) + \Delta(\psi_\uparrow^\dagger \psi_\downarrow^\dagger + \psi_\downarrow \psi_\uparrow) \Big)\, dx$$

$$= \frac{i}{2} \int \eta^T \big(\Gamma \partial + M(x)\big) \eta \, dx, \qquad \text{where} \quad \eta = \begin{pmatrix} \psi_\uparrow + \psi_\uparrow^\dagger \\ -i(\psi_\uparrow - \psi_\uparrow^\dagger) \\ \psi_\downarrow + \psi_\downarrow^\dagger \\ -i(\psi_\downarrow - \psi_\downarrow^\dagger) \end{pmatrix}.$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 & h_x - \Delta \\ 0 & 0 & -h_x - \Delta & \\ 0 & h_x + \Delta & 0 & 0 \\ -h_x + \Delta & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Pf}(M(x)) = \Delta^2 - h_x^2$ changes sign

$\Rightarrow$ **zero mode**



Dirac Hamiltonian in dimension d

- Hamiltonian:

$$\widehat{H} = \frac{i}{2}\int \eta^T \left(\Gamma^\alpha \partial_\alpha + M\right)\eta \, d^d x.$$

real symmetric

real skew-symmetric

- Physical requirement:

  The spectrum $\varepsilon(\vec{q}) = \text{eigenvalues}\left(-\Gamma^\alpha q_\alpha + iM\right)$ should be gapped.

  – Nontrivial mass terms: $\Gamma^\alpha M = -M\Gamma^\alpha$

    $\varepsilon(\vec{q})^2 = \text{e.v.}(q^2 + M^2) > 0$ — the spectrum *is* gapped.

  – Trivial mass terms: $\Gamma^\alpha M = M\Gamma^\alpha$. Such terms do not open a gap and can be removed by a homotopy.

Dirac operators with symmetries

- Canonical form (achieved by a homotopy):
  - Clifford symmetries: $e_1^2, \ldots, e_q^2 = -1$ (fixed)
  - Dirac matrices: $(\Gamma^\alpha)^2 = 1, \quad \Gamma^\alpha e_j = -e_j \Gamma^\alpha$ (fixed)
  - *Mass term (the object to classify):*

$$M^2 = -1, \qquad \text{where} \quad \Gamma^\alpha M = -M\Gamma^\alpha$$

- This is the Clifford extension problem:

$$\mathrm{Cliff}^{d,q} \setminus \mathrm{Cliff}^{d,q+1} \sim \mathrm{Cliff}^{p-d,0} \setminus \mathrm{Cliff}^{p-d+1,0}, \qquad \text{where } p = q + 2.$$

Solution: $KO^{d-p}(\text{point}) = \pi_0(R_{p-d})$.

- Dimension shift: *Each spactial dimension is equivalent to a positive Clifford generator and cancels a negative generator.*

- On a spin manifold X, mass terms are classified by $KO^{d-p}(X)$.

The periodic table understood:

p	$\pi_0(R_p)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$	"wrong T": $T^2=1$	No symmetry: p_x+ip_y (SrRu)	T (^{3}He-B)
1	$\mathbb{Z}_2$	No symmetry (Majorana chain)	T-invariant triplet superconductors $((p_x+ip_y)\uparrow+(p_x-ip_y)\downarrow)$	T and Q (BiSb)
2	$\mathbb{Z}_2$	T-inv. triplet superconductors ($(TMTSF)_2X$)	T and Q: T-invariant insulators (HgTe)	
3	0			
4	$\mathbb{Z}$			

Band insulators and K-theory on the torus

The momentum space is $\overline{\mathbb{T}}^d$, which means that we consider complex vector bundles on the torus $\mathbb{T}^d$ with the involution $q \to -q$, which is accompanied by an antiliner map in the fiber.

$$KO^{-p}(\overline{\mathbb{T}}^d) \cong KO_p(\mathbb{T}^d) \cong KO^{d-p}(\mathbb{T}^d) \cong$$

$$\cong \underbrace{KO^{d-p}(\text{point})}_{\substack{\text{like for Dirac operators} \\ \text{or disordered systems}}} \oplus \underbrace{\widetilde{KO}^{d-p}(\text{point})}_{\substack{\bigoplus_{k=0}^{d-1} \binom{d}{k} KO^{k-p}(\text{point}) \\ \text{(non-canonical decomposition)}}}$$

Topology of Hamiltonians in the momentum space

Baum-Connes isomorphism (K-theory analog of the Fourier transform)

<u>Example: 3D insulators with time-reversal symmetry</u>

$$KO^{-4}(\overline{\mathbb{T}}^3) \cong \mathbb{Z}_2 \oplus 3\mathbb{Z}_2$$

Distinguishes strong topological insulators

Interactions

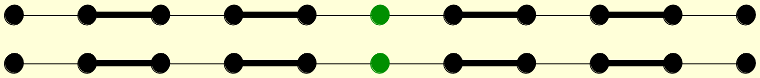
symmetry dimension

- Free fermion phases: $F(G, d)$ (the present classification)
- Interacting fermion phases: $I(G, d)$ (future classification)
- "Free" is a special case of "interacting":

$$\zeta : F(G, d) \to I(G, d).$$

  - *Is ζ surjective?* **No,** there are interacting phases that have no free analogue (e.g. FQHE).
  - *Is ζ injective?* A counterexample would be two distinct free phases, X and Y, that can be continuously connected through an interacting phase. **Such an example exists!**
- Nevertheless, in some symmetry settings the free classification is stable, eg. $G = U(1)$, $d = 2$ (IQHE).

The counterexample



Majorana chains: $c_{j,\alpha}$

site number   chain

$$T(c_{j,\alpha}) = (-1)^j c_{j,\alpha}, \qquad \text{(unusual: } T^2 = 1\text{)}$$

- This symmetry prohibits terms like $ic_{j,\alpha}c_{k,\beta}$, where j and k have the same parity. Therefore unpaired modes on parallel chains do not cancel.

- The noniteracting classification predicts an integer invariant (i.e., $F = \mathbb{Z}$).

$$\nu = (\#\text{ of solitons at even } j) - (\#\text{ of solitons at odd } j)$$

- Yet 8 solitons can cancel each other, and the corresponding dimer phases on 8 parallel chains can be continuously connected through an interacting phase! ($I = \mathbb{Z}_8$)

Summary

- Topological phases of noninteracting fermions can be classified using K-theory.

- Most natural symmetries are captured by complex and real Clifford algebras ($2+8$ universality classes).

- Dimension change is described by periodic shift $\pmod 2$ or $\pmod 8$ — Bott periodicity.

- The basic description uses Dirac operators, while a more general theory involves analytic K-homology.

- In some cases (IQHE, Majorana chains, $p_x + ip_y$, TRI insulators in 2d and 3d) the classification is stable to strong interactions.

- That is not true in general; a counterexample is known.