Classical and Quantum Duality

Paul Fendley

Kramers-Wannier duality of two-dimensional classical lattice models and the ”topological symmetry” of interacting anyonic chains are the same symmetry.

This has applications to both classical and quantum lattice models.

In classical models, this allows us to generalize duality to essentially any lattice model with an integrable critical point.

In quantum models, this allows one to find ground states with non-abelian topological order, and moreover construct relatively simple local Hamiltonians with such ground states.

What is Kramers-Wannier duality?

The classical Q-state Potts model has a spin $\sigma_i = 1..Q$ at each site i of a lattice. The energy is simply

$$E = -J \sum_{<ij>} \delta_{\sigma_i \sigma_j}$$

so the partition function is

$$Z_L(K) = \sum_{\{\sigma_i\}} e^{K \sum_{<ij>} \delta_{\sigma_i \sigma_j}} \; .$$

where $K \equiv J/T$ and L indicates the lattice the spins live on.

Lots of interesting structure in this model: critical point in 2d for $Q \leq 4$; Temperley-Lieb algebra; Tutte polynomial,. . .

Perhaps the most remarkable thing about the Potts model is its (Kramers-Wannier) duality.

Duality is the very non-obvious statement that with appropriate boundary conditions,

$$Z_L(K) = Z_{\widehat{L}}(\widetilde{K})$$

where $\widehat{L}$ is the dual lattice to L, while K and the dual coupling $\widetilde{K}$ are related by

$$(e^K - 1)(e^{\widetilde{K}} - 1) = Q \, .$$

The self-dual point is at $K = K_c$, where $(e^{K_c} - 1) = \sqrt{Q}$. In general, the duality maps between a model in the “high-temperature” phase $K < K_c$ and one in the “low-temperature” (ordered) phase $K > K_c$.

In terms of domain walls, the weight for a link separating spins $\sigma_i = a$, $\sigma_j = b$ is

$$(1 - \delta_{ab}) \qquad + \qquad e^{K} \delta_{ab} \; .$$

In the dual model, the analogous weight is

$$(1 - \delta_{\alpha\beta}) \qquad + \qquad e^{\widetilde{K}} \delta_{\alpha\beta}$$

where $\mu_i = \alpha$ and $\mu_j = \beta$ are the values of the spins in the dual model.

Build up the transfer matrix by taking products of these, for both horizontal and vertical links. Think in terms of the domain walls, so each link is a two-state system: either the domain wall is there, or it isn't.

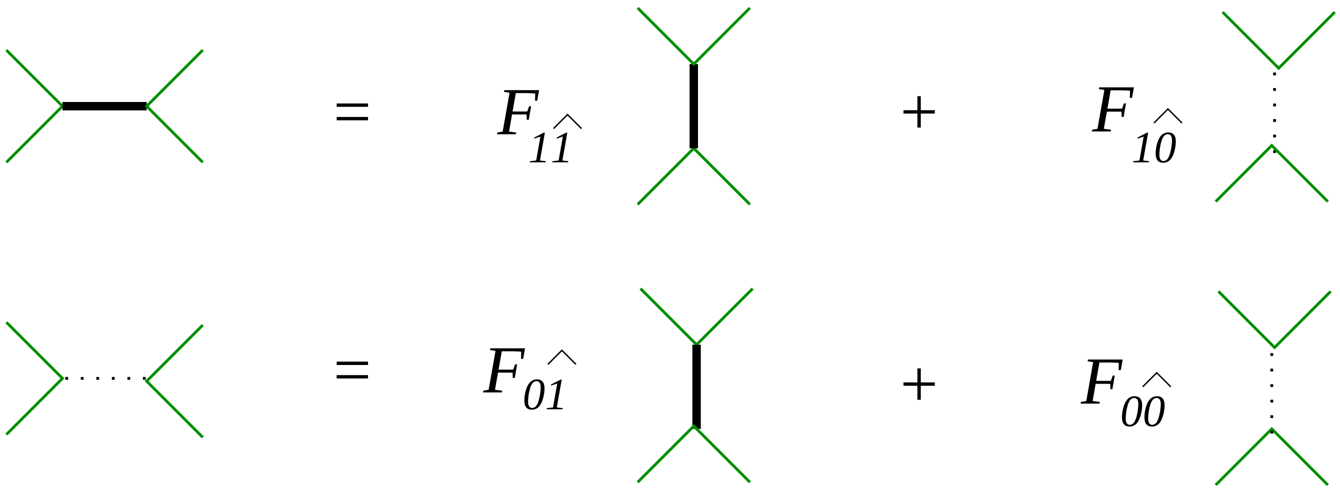
Using $(e^K - 1)(e^{\widetilde{K}} - 1) = Q$ relates the couplings for each state of the two-state system and the dual two-state system

$$\begin{pmatrix} e^{\widetilde{K}} \\ 1 \end{pmatrix} \propto \begin{pmatrix} 1 & Q-1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{K} \\ 1 \end{pmatrix}$$

Everything can be generalized to non-integer Q by using the Temperley-Lieb algebra; note that $(1 - \delta_{ab})$ and δ_{ab} are (orthogonal) projectors. Then the TL parameter d (the weight of each closed loop) obeys $d = \sqrt{Q}$.

Topological aficionados note: this “duality matrix” changing the two bases is precisely the $SU(2)_k$ four-anyon fusion matrix for $\sqrt{Q} = 2\cos(\pi/(k+2))$!

Moreover, the standard pictures even look similar:



So what does duality have to do with non-abelian anyons?

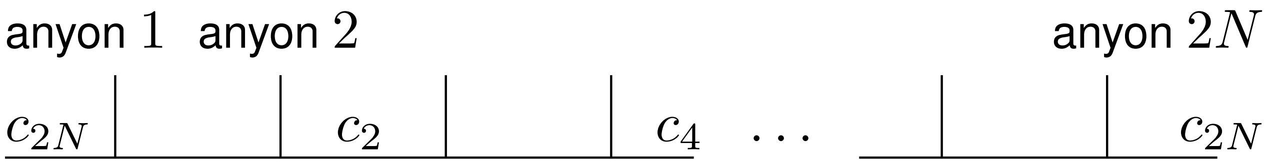
Duality = topological symmetry

Topological symmetry is by construction a symmetry of interacting anyonic chains.

Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang and Freedman

The degrees of freedom in an interacting anyon chain are the fusion channels.

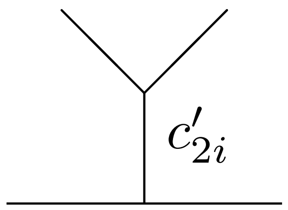
For Ising fusion rules $\sigma \times \sigma = I + \psi; \quad \sigma \times \psi = \sigma$; the fusion channels of $2N$ anyons are specified by the picture (unlabeled links are σ)



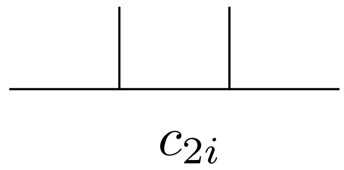
This is an Ising Hilbert space! Each c_{2i} can be either I or ψ, while each $c_{2i-1} = \sigma$. Thus we have an Ising chain with $\sigma_i = +1$ when $c_{2i} = I$, while $\sigma_i = -1$ corresponds to $c_{2i} = \psi$.

The Hamiltonian of an interacting anyonic chain is then defined by demanding that it commute with “adding a loop”.

The Potts F matrix acts as



$$= \sum_c F_{c_{2i'} c_{2i}}$$

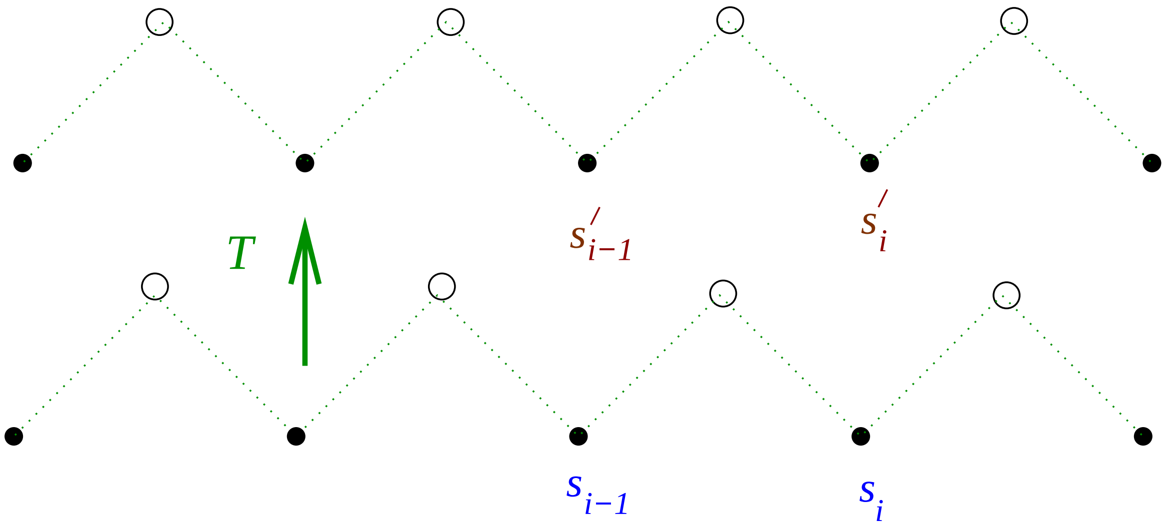


where as before all unlabeled line segments are σ.

By particular combination of F matrix moves, one can convert the process of adding a loop into an operation on the Hilbert space

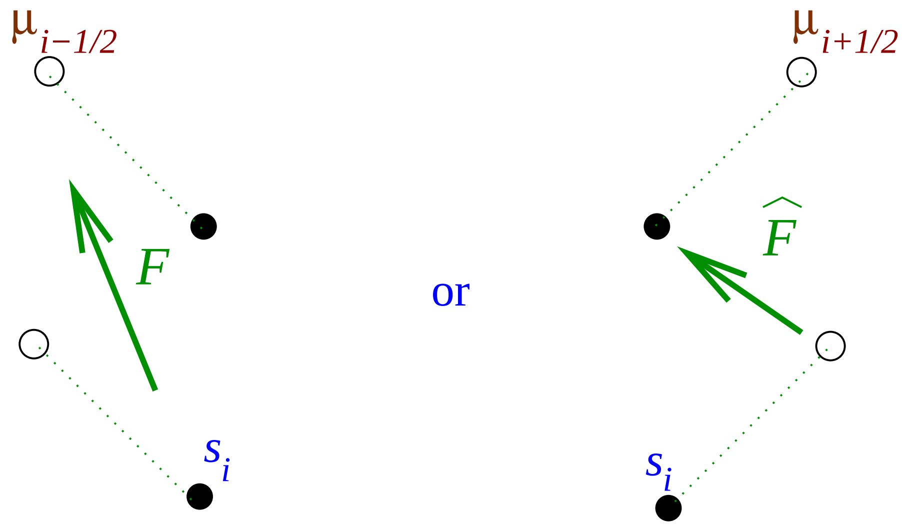
Levin and Wen; California stars

The topological symmetry is easiest to see in the classical model. The Potts transfer matrix acts on a Q^N-dimensional space of spins, the same Hilbert space as the 1d quantum Hamiltonian:



In this picture, the domain walls are either horizontal or vertical lines connecting the sites on the dual lattice. The dual sites correspond to $c_{2i-1} = \sigma$ channels on the chain. For more general models, need to have varying degrees of freedom here.

The F matrix in the lattice model acts as:

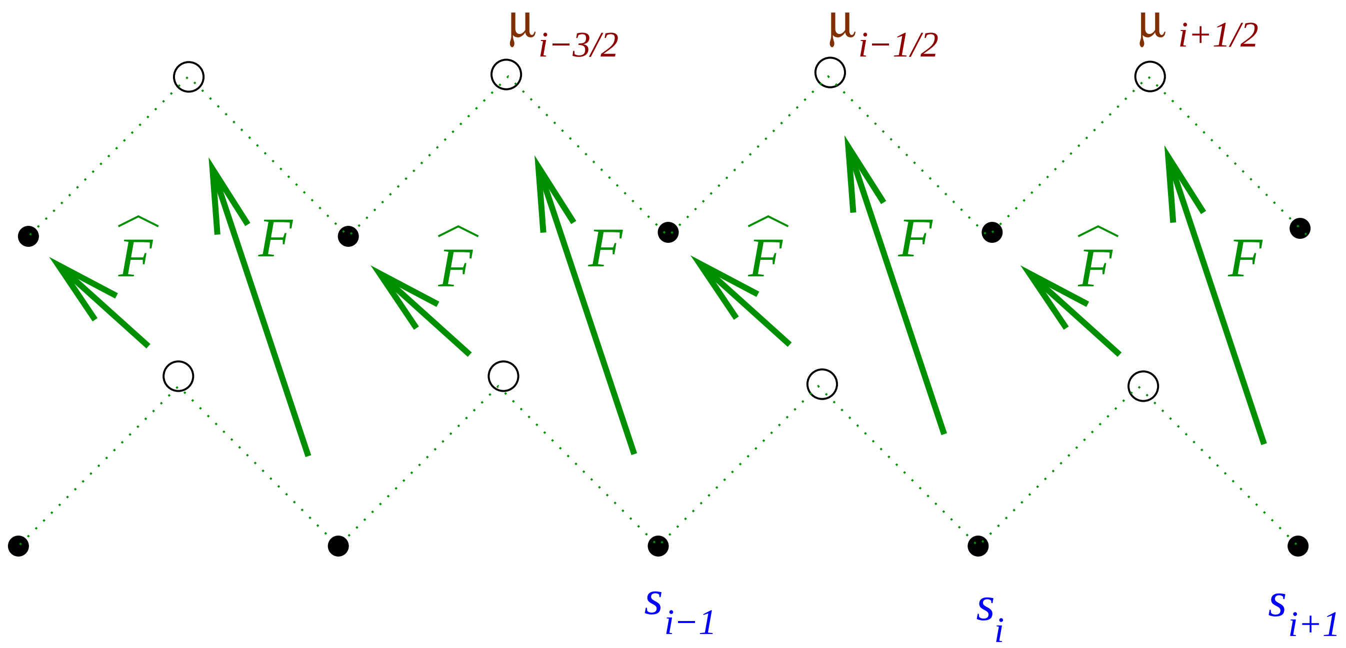


The arrow indicates that F or $\widehat{F}$ can be thought of as changing the value of the spin at the bottom of the arrow to the value at the top, depending on the values of the other two spins.

For Ising,

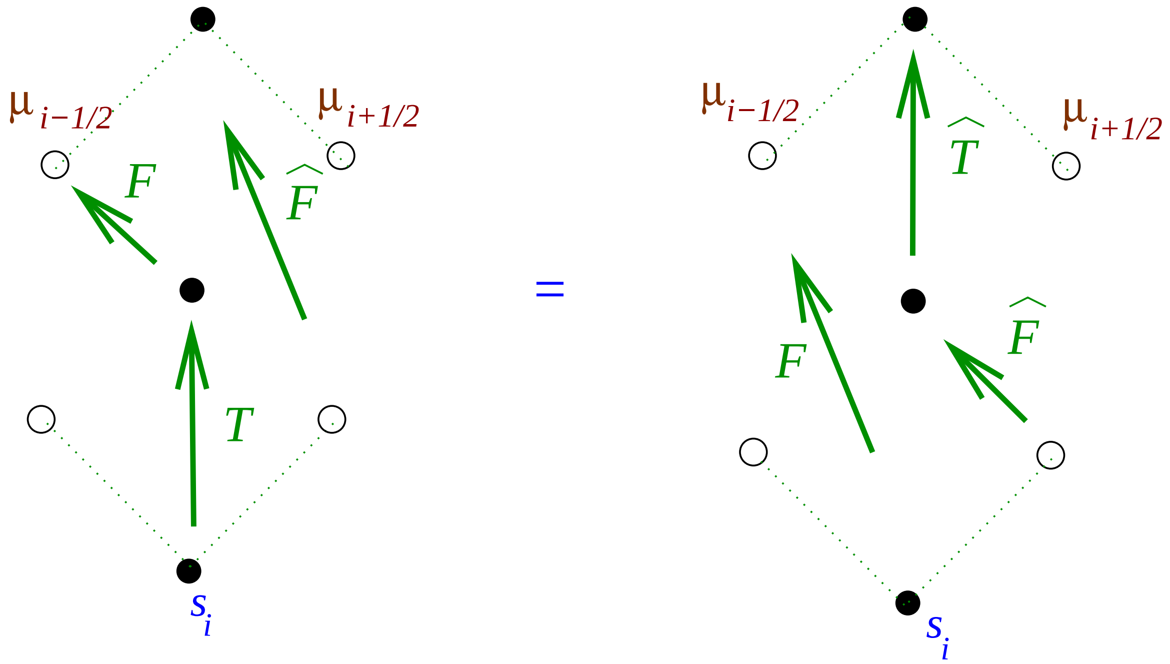
$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \widehat{F} = (-1)^{\delta_{s_i \mu_{i+1/2}}}$$

The topological symmetry generator in the classical lattice model is then simply



Note that the the non-trivial states shift; they are now on the dual lattice!

To prove that this is a symmetry, it needs to “commute” with the transfer matrix. More precisely, when commuted, it needs to give the dual transfer matrix:



Integrability aficionados will recognize this as the Yang-Baxter equation. Up to (important) phases which cancel in the YBE, F is the limit of T with infinite spectral parameter.

This construction enables one to generalize duality to many critical integrable models. Integrable lattice models are known which in the continuum limit yield all CFTs

$$\frac{G_k \times G_l}{G_{k+l}}$$

In fact, there can be even more than one duality for a given CFT. For example, for each minimal CFT, there is both a Potts duality (what the California stars study) and a tricritical Potts duality.

Once you have the duality, there is no need to demand integrability: you can e.g. study how perturbations and operators in the lattice model and the CFT behave under the duality.

This duality is also useful when studying 2d quantum nets.

Quantum self-duality allows the construction of a simple(r) Hamiltonian annihilating a topologically-ordered ground state consisting of a sum over nets.

For the square lattice, it requires only four-spin interactions.

To build a quantum loop/net model,

1. find a 2d classical loop model where loops are critical but local degrees of freedom are not (e.g. percolation)

2. use each loop configuration as a basis element of the quantum Hilbert space

3. find a Hamiltonian whose ground state a sum over loop configurations with the appropriate weighting, so that

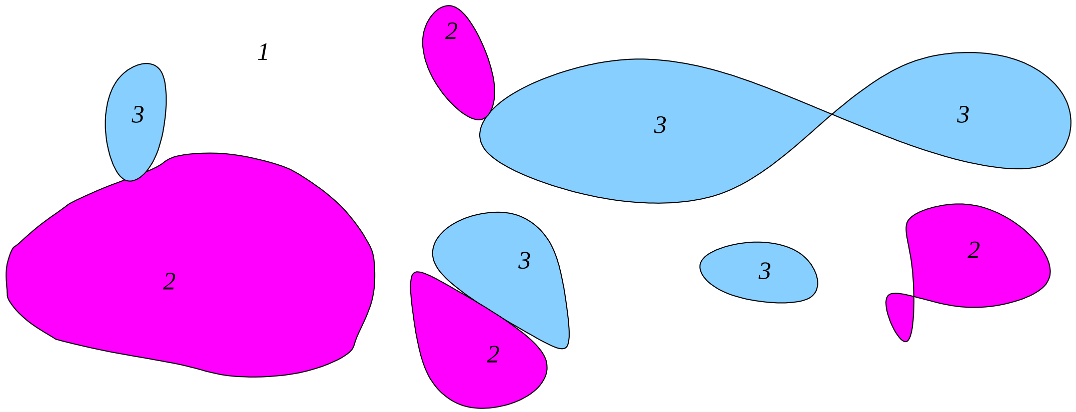
4. if you "cut" a loop, you end up with two deconfined anyonic excitations

Kitaev; Moessner and Sondhi; Freedman

The quantum Potts model uses the 2d classical Potts model for its underlying degrees of freedom.

The domain wall on each link forms a two-state quantum system, i.e. the presence of the domain wall is one state $|1\rangle$, while the absence is the orthogonal state $|0\rangle$.

Each domain-wall configuration is a net: by construction, it has no ends.



The Hilbert space of the quantum model for N links is still fully 2^N-dimensional.

Configurations with “net ends” are by construction anyonic excited states.

The weight of each net $|N\rangle$ in the ground state $|\Psi\rangle$ of the quantum model is found from the classical model:

$$\langle N|\Psi\rangle = (Q-1)^{-L_N}\chi_{\widehat{N}}(Q)$$

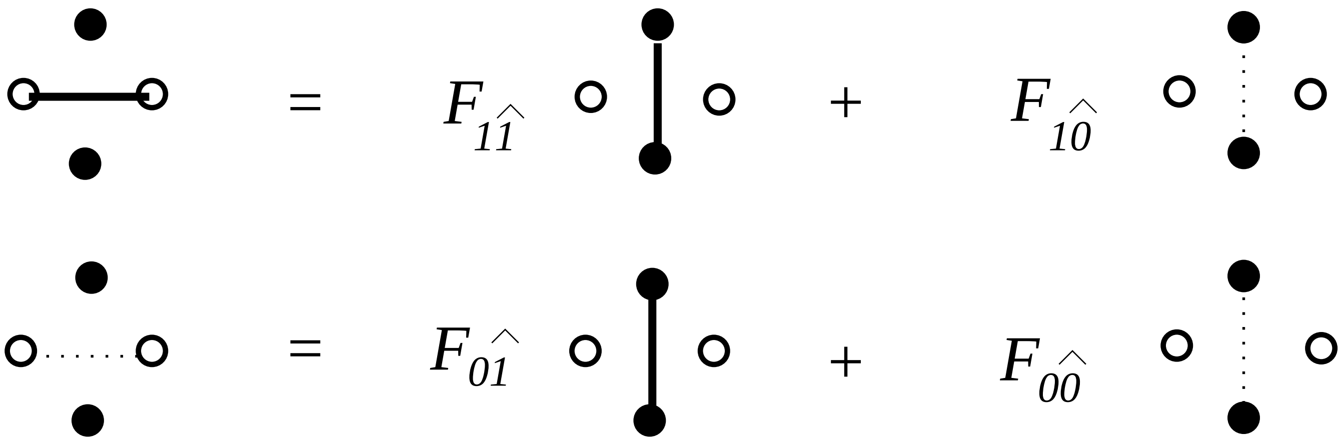
Fendley and Fradkin

$\chi_{\widehat{N}}(Q)$ is the number of spin configurations allowed for each domain-wall configuration N. $\chi_{\widehat{N}}$(Q) is called the chromatic polynomial, and depends only on the topology of N. Its definition can be extended to all Q, not just integers.

L_N is the "length" of each net, the number of $|1\rangle$ states. This factor is put there to ensure that...

The quantum Potts model is quantum self-dual.

This means we can equivalently define the model in terms of dual domain walls, $|\widehat{0}\rangle$, $|\widehat{1}\rangle$.



$$F = \begin{pmatrix} \langle\widehat{0}|0\rangle & \langle\widehat{0}|1\rangle \\ \langle\widehat{1}|0\rangle & \langle\widehat{1}|1\rangle \end{pmatrix} = \frac{1}{\sqrt{Q}} \begin{pmatrix} 1 & \sqrt{Q-1} \\ \sqrt{Q-1} & -1 \end{pmatrix}$$

The weight of each dual net $|D\rangle$ in the ground state is

$$\langle D|\Psi\rangle = \left(\frac{1}{\sqrt{Q-1}}\right)^{L_D} \chi_{\widehat{D}}(Q)$$

This is the same ground state $|\Psi\rangle$ in a new basis!

This quantum self-duality is highly non-obvious, and extremely useful.

A Hamiltonian H with $|\Psi\rangle$ a ground state can be found simply by demanding that H annihilate all states which are not nets and annihilate all states which are not dual nets.

For the square lattice:

$$\begin{aligned} H &= \sum_{+} [P_1 P_0 P_0 P_0 \; + \; \text{rotations}] \\ &\quad + \sum_{\square} [P_{\hat{1}} P_{\hat{0}} P_{\hat{0}} P_{\hat{0}} \; + \; \text{rotations}] \end{aligned}$$

where P_i projects onto the states $|i\rangle$, and $P_{\hat{i}} = F P_i F$.

Conclusions

It has long been known that integrability and topological order are related: integrable models yield essentially all the known knot polynomials. Now we see that even old-school Kramers-Wannier duality has a topological interpretation.

Thinking about topology leads to a natural extension of self-duality to two-dimensional quantum systems.

This in turn leads to a much simpler Hamiltonian annihilating topologically-ordered ground states.