Topological order at finite temperatures

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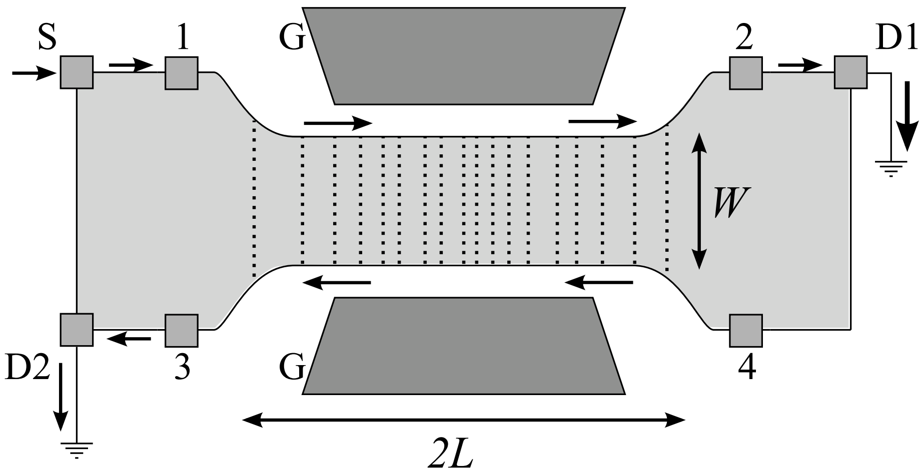
Interferometer section ...

Bas Overbosh & C^3 (arXiv:0810.1289)

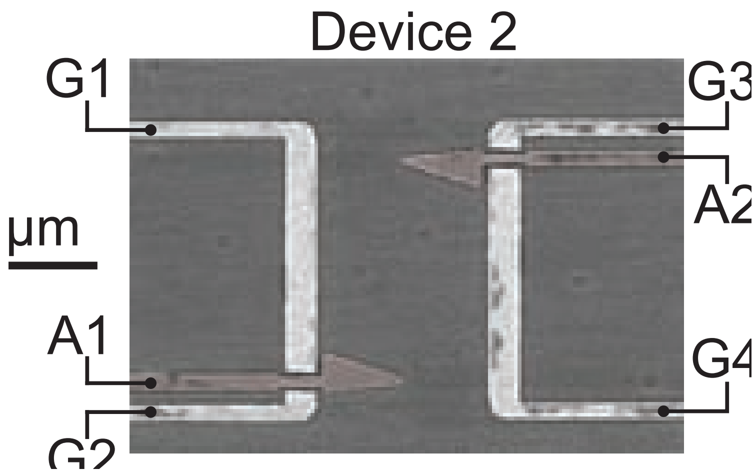


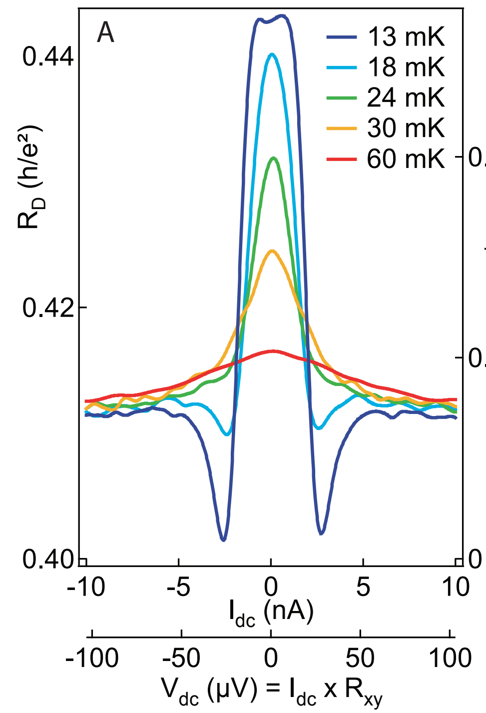
$$\frac{eV_{\rm res}}{\hbar} = \frac{vW}{\ell_B^2}$$

$$v_{\rm min} \simeq \frac{2\pi}{(\frac{e^*}{e})(\frac{W}{\ell_B})} \frac{k_B T}{\hbar} \ell_B$$

$$v_{\rm min} \simeq 25 m/s$$

Radu et al (2008)



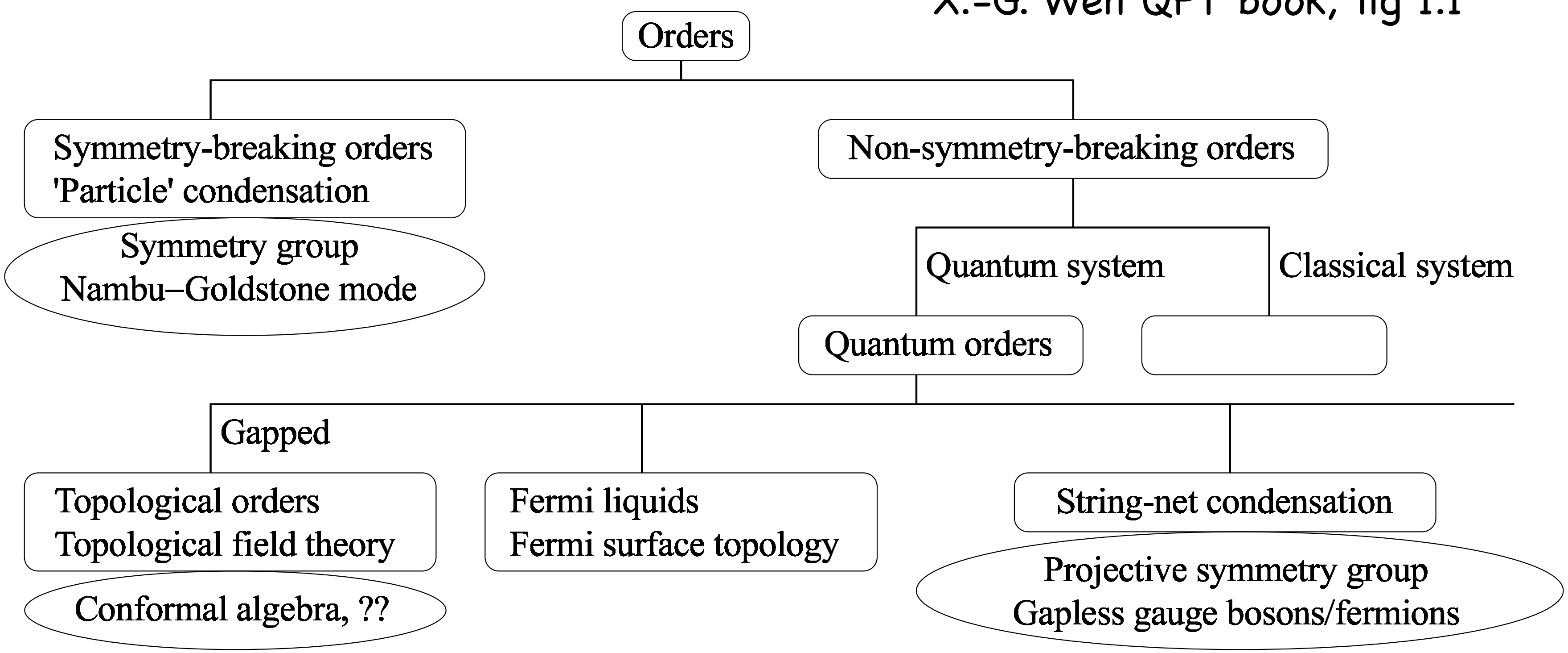


- What is topological order at $T \neq 0$?
- What is the classical counterpart of quantum topological order ?

- How to define a measure of either ?

- What is topological order at T≠0 ?
- What is the classical counterpart of quantum topological order ?

X.-G. Wen QFT book, fig 1.1



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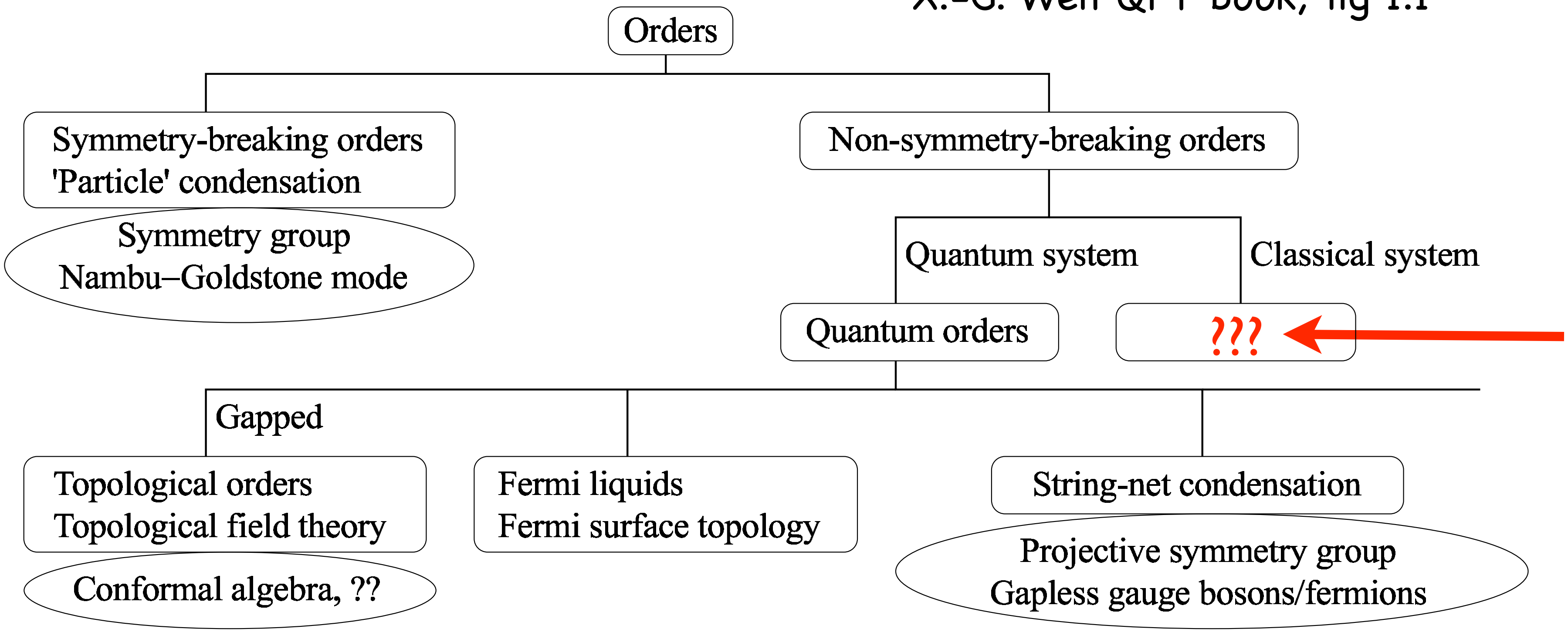
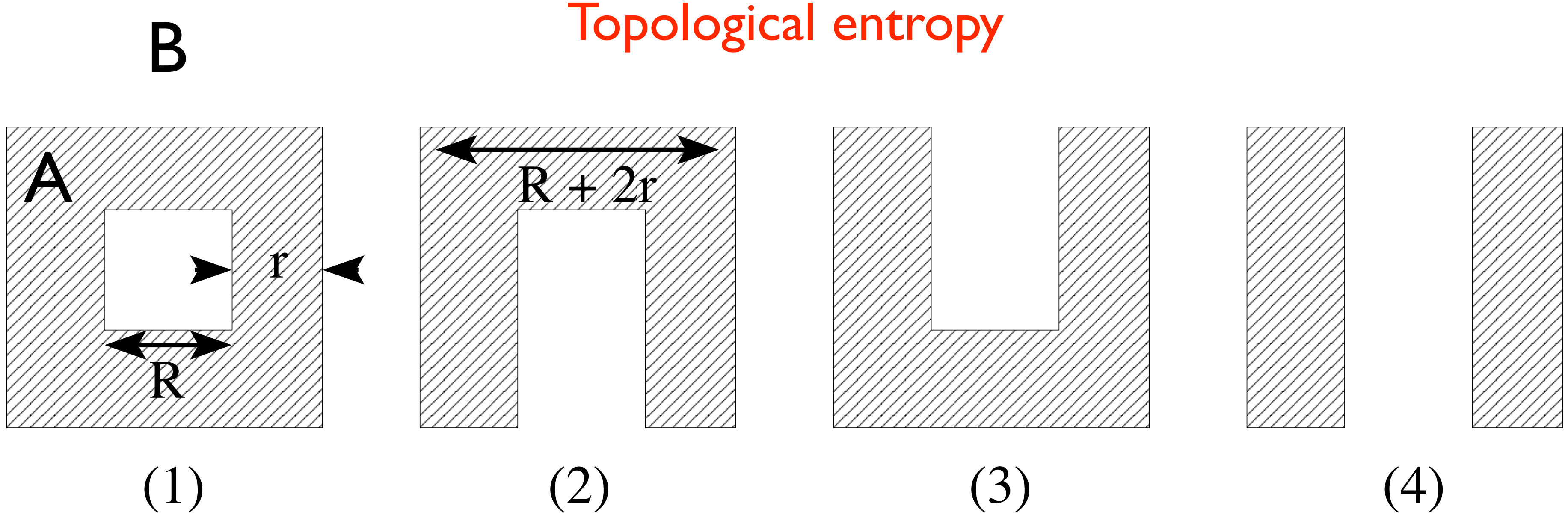


- How to define a measure of either ?

T=0 (still)

Levin & Wen, Kitaev & Preskill (2006)

Topological entropy



von Neumann entropy

Given a pure state density matrix

$$\hat{\rho} = |\Psi_{\rm GS}\rangle\langle\Psi_{\rm GS}| \;\rightarrow\; \hat{\rho}_A = {\rm tr}_B\, \hat{\rho} \;\rightarrow\; S_A = -{\rm tr}_A\, (\hat{\rho}_A \, \ln \hat{\rho}_A)$$

$$S_A = S_B = \alpha\, \mathcal{S}_{AB} - \gamma_{AB} + \ldots$$

$$S_{\rm topo} = -S_{1A} + S_{2A} + S_{3A} - S_{4A}$$

T≠0

T=0: → **Finite T:**

$$\hat{\rho} = |\Psi_{\rm GS}\rangle\langle\Psi_{\rm GS}|$$

$$\hat{\rho}(T) = \frac{1}{Z}\, e^{-\beta \hat{H}}$$

$$\hat{\rho}_A = \mathrm{tr}_B\ \hat{\rho}$$

$$S_A = -\mathrm{tr}_A\left(\hat{\rho}_A\ \ln\hat{\rho}_A\right)$$

$$S_A = S_B$$

$$S_A \neq S_B$$

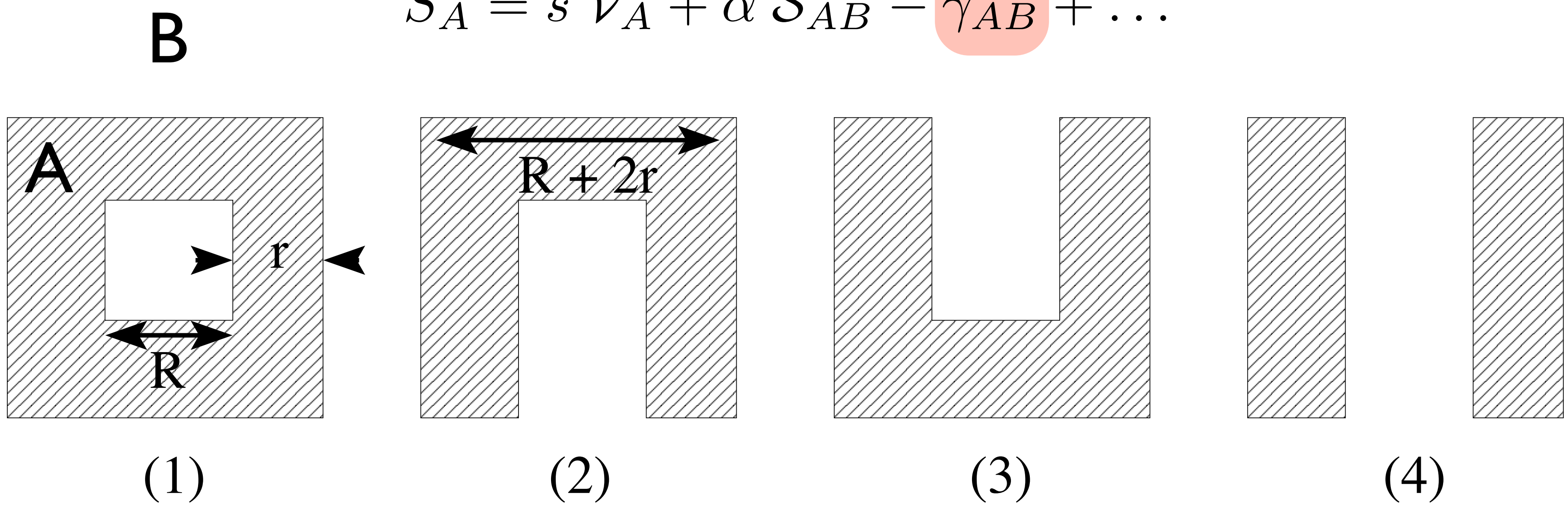
$$S_A = S_B = \alpha\,\mathcal{S}_{AB} - \gamma_{AB} + \ldots$$

$$S_A = s\,\mathcal{V}_A + \alpha\,\mathcal{S}_{AB} - \gamma_{AB} + \ldots$$

Topological piece within T≠0 equilibrium density matrix

Castelnovo & C[3] (2007)

$$S_A = s\,\mathcal{V}_A + \alpha\,\mathcal{S}_{AB} - \gamma_{AB} + \ldots$$



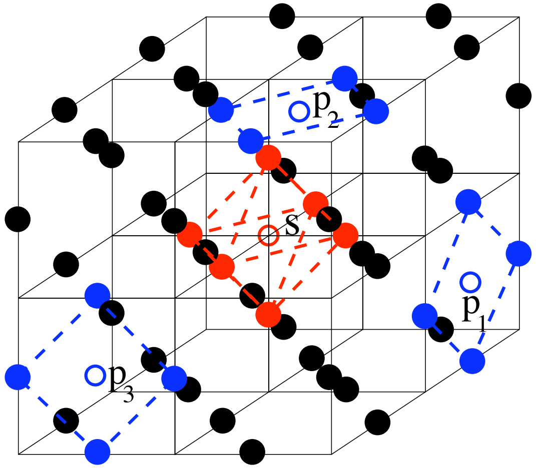
Volume and surface terms do cancel (1-2-3+4)

$$S_{\text{topo}} = -S_{1A} + S_{2A} + S_{3A} - S_{4A}$$

Case study: toric code in 2D, 3D, ...

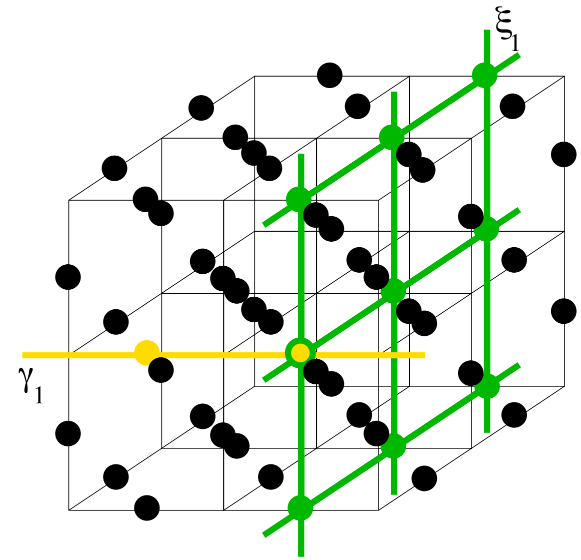
Kitaev (1997)
Hamma, Zanardi, and Wen (2005)

$$H = -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p$$



$$\Gamma_1 = \prod_{i \in \gamma_1} \sigma_i^z \qquad \Gamma_2 = \prod_{i \in \gamma_2} \sigma_i^z \qquad \Gamma_3 = \prod_{i \in \gamma_3} \sigma_i^z$$

$$\Xi_1 = \prod_{i \in \xi_1} \sigma_i^x \qquad \Xi_2 = \prod_{i \in \xi_2} \sigma_i^x \qquad \Xi_3 = \prod_{i \in \xi_3} \sigma_i^x$$



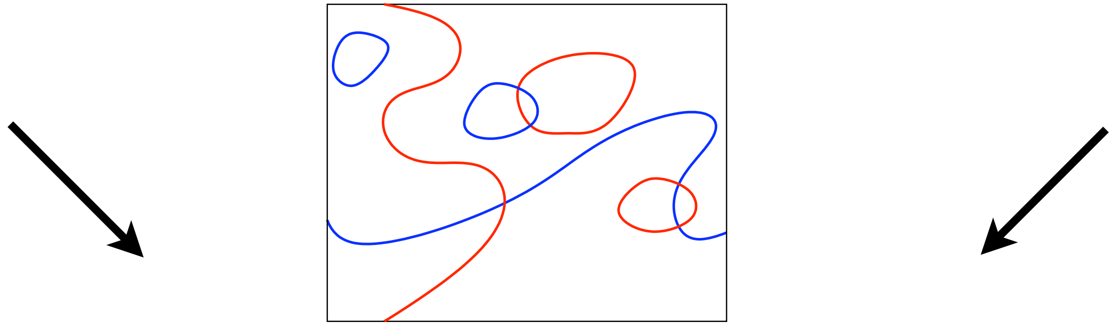
In any D ...

The contributions to the entropies from the two terms (plaquettes and stars) in the Hamiltonian are additive

$$S_{\rm VN}(\mathcal{A};T) = S_{\rm VN}^{(S)}(\mathcal{A};T/\lambda_A) + S_{\rm VN}^{(P)}(\mathcal{A};T/\lambda_B)$$

$$S_{\rm topo}(T) = S_{\rm topo}^{(S)}(T/\lambda_A) + S_{\rm topo}^{(P)}(T/\lambda_B).$$

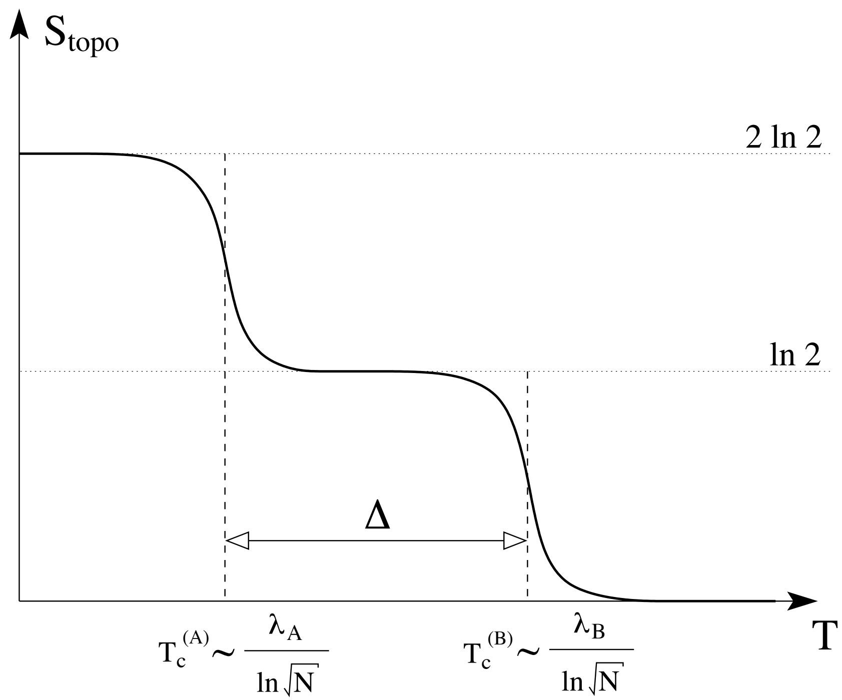
$$\begin{cases} \lambda_A \to 0 \\ \lambda_B \to \infty \end{cases}$$



$$\begin{cases} \lambda_A \to \infty \\ \lambda_B \to 0 \end{cases}$$

The non-vanishing quantum topological entropy arises from the plaquette and star terms in the Hamiltonian as two equal and independent (i.e., classical) contributions

In 2D

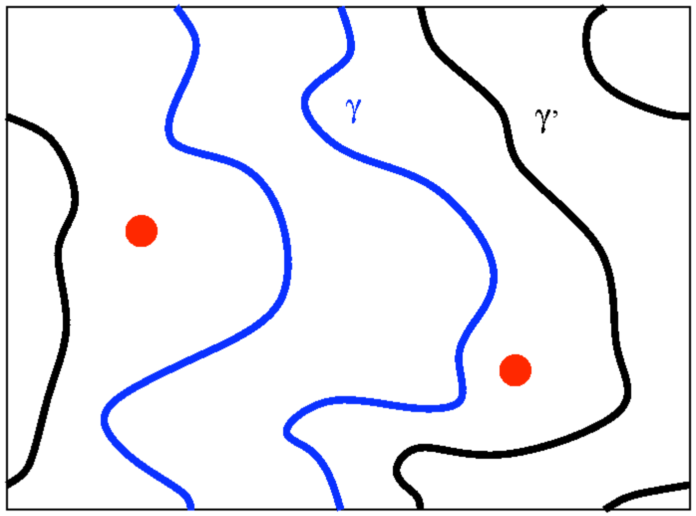


Entropy vanishes when the number of defects on a system of size N is O(1)

$$N e^{-\lambda_{A/B}/T} \sim 1$$

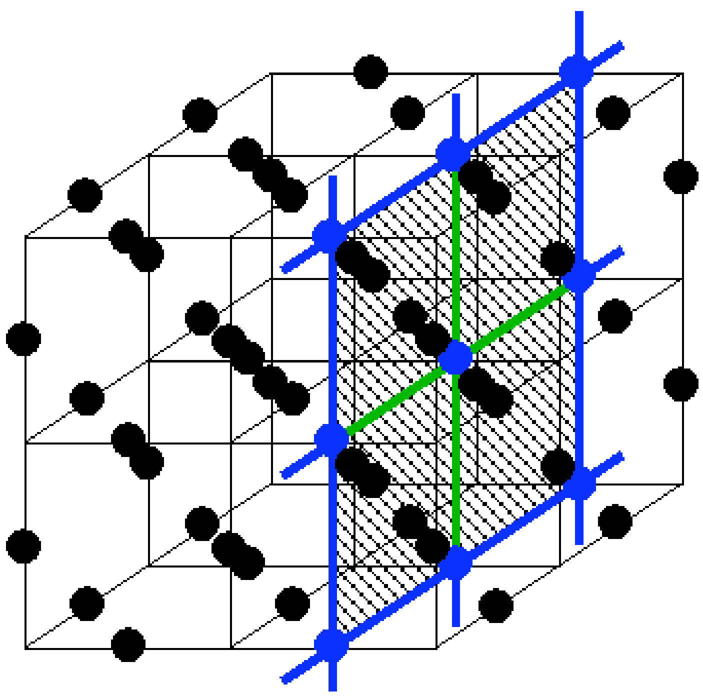
Two (deconfined) defects immediately spoil the order parameter $\Gamma_1 = \prod_{i \in \gamma_1} \sigma_i^z$

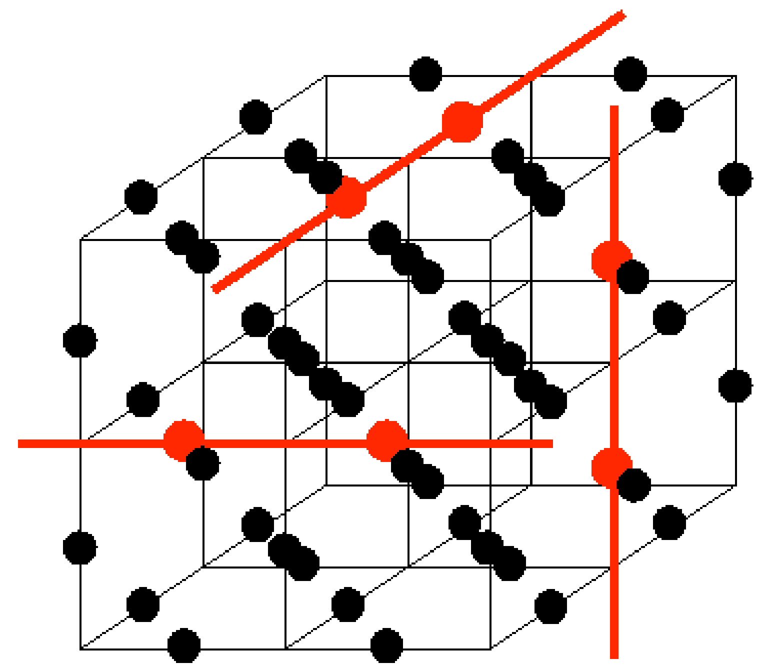
order only if NO defects at all!



In 3D

Non-local operators distinguishing the different sectors are winding loops and winding membranes





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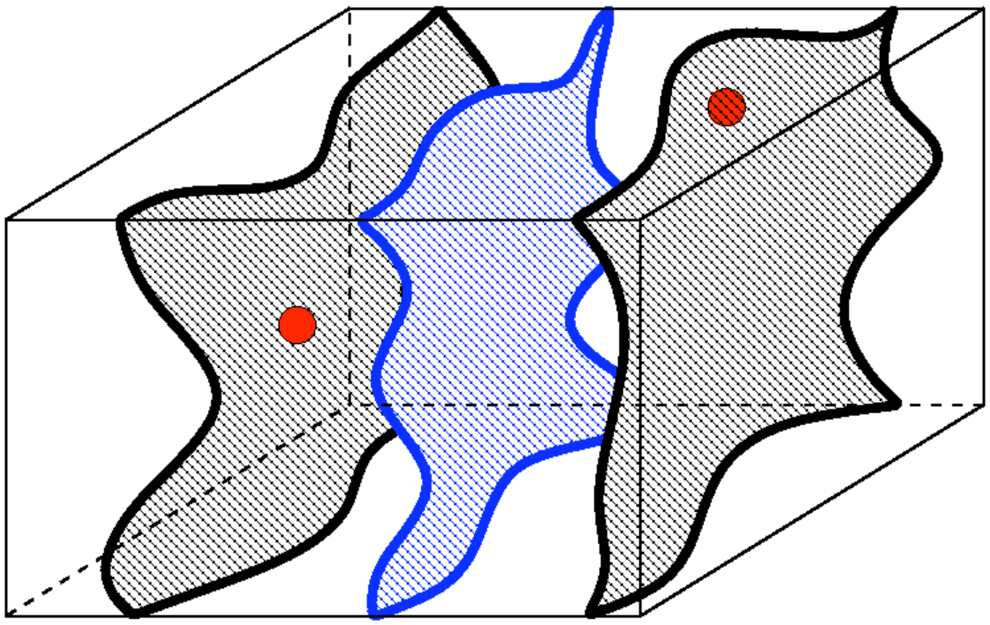
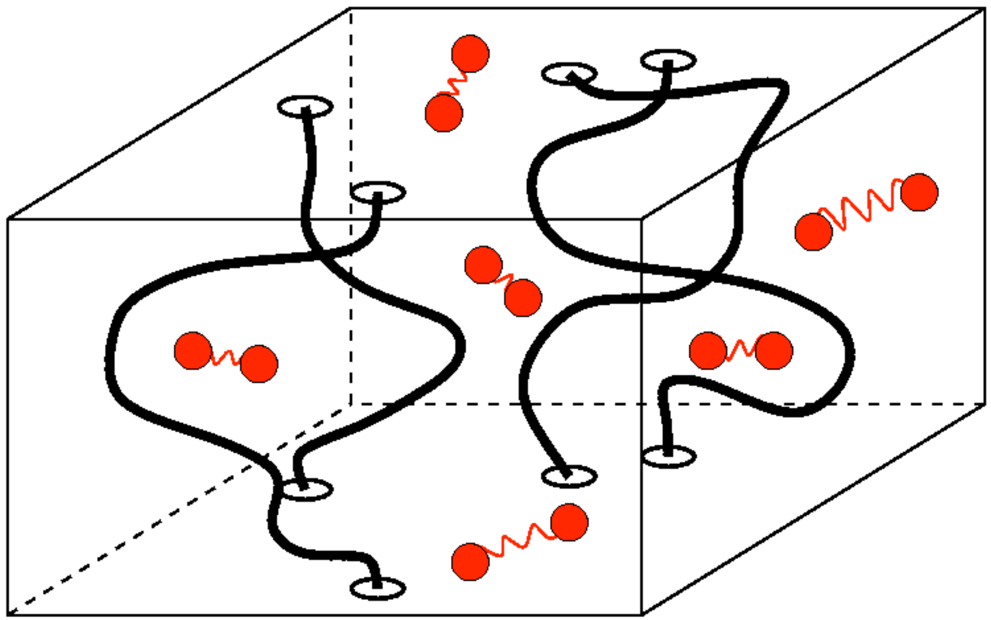


fragile



ROBUST!

In 3D

Non-local operators distinguishing the different sectors are
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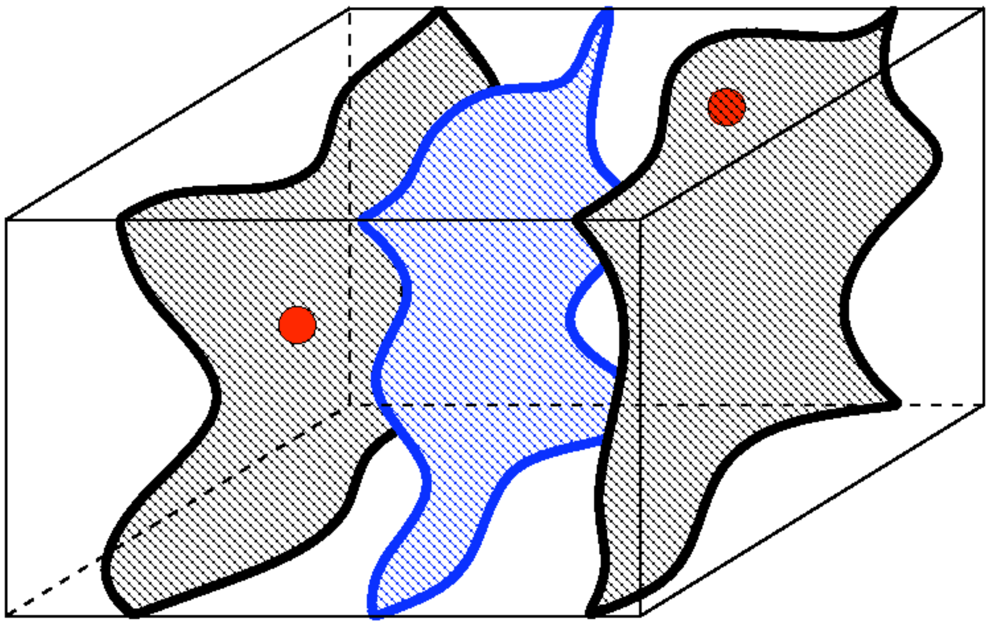

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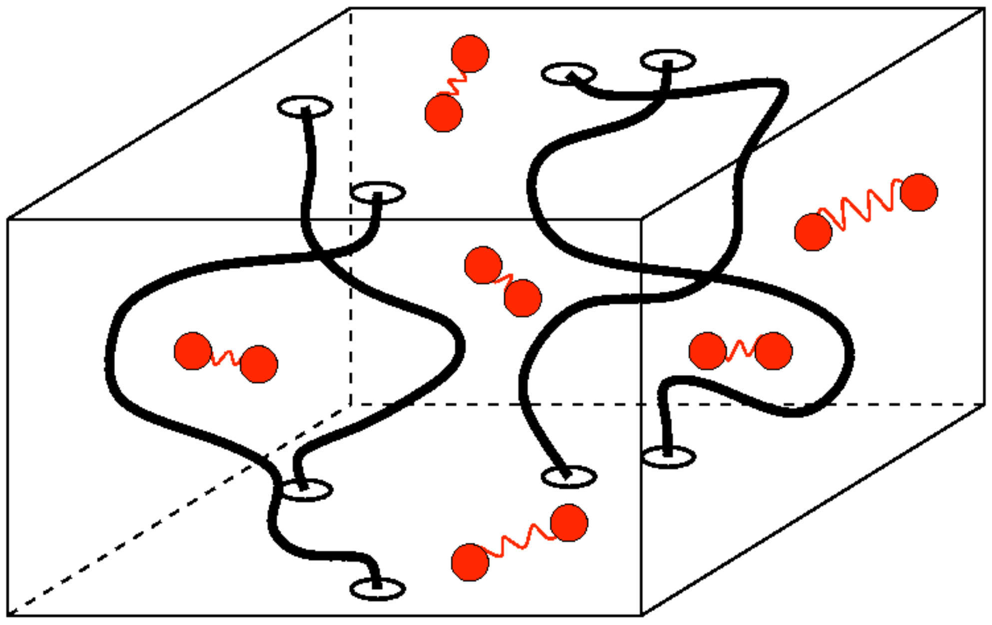
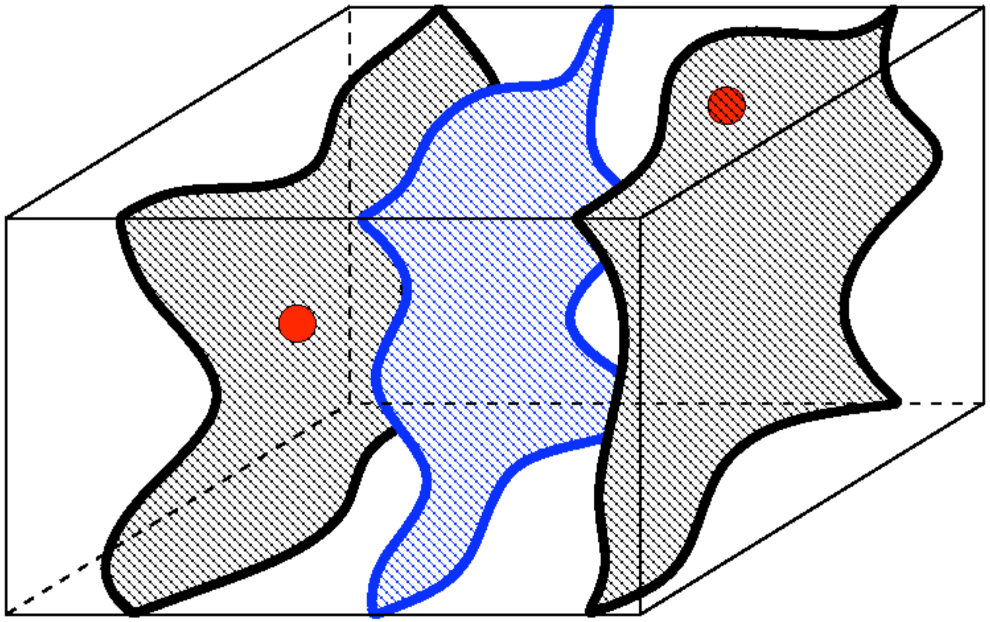


ROBUST!

$$\Gamma_1 = \langle \prod_{i \in \gamma_1} \sigma_i^z \rangle \rightarrow \frac{1}{N_{\gamma_1}} \sum_{\gamma_1} \langle \prod_{i \in \gamma_1} \sigma_i^z \rangle$$

In 3D

Non-local operators distinguishing the different sectors are winding loops and winding membranes



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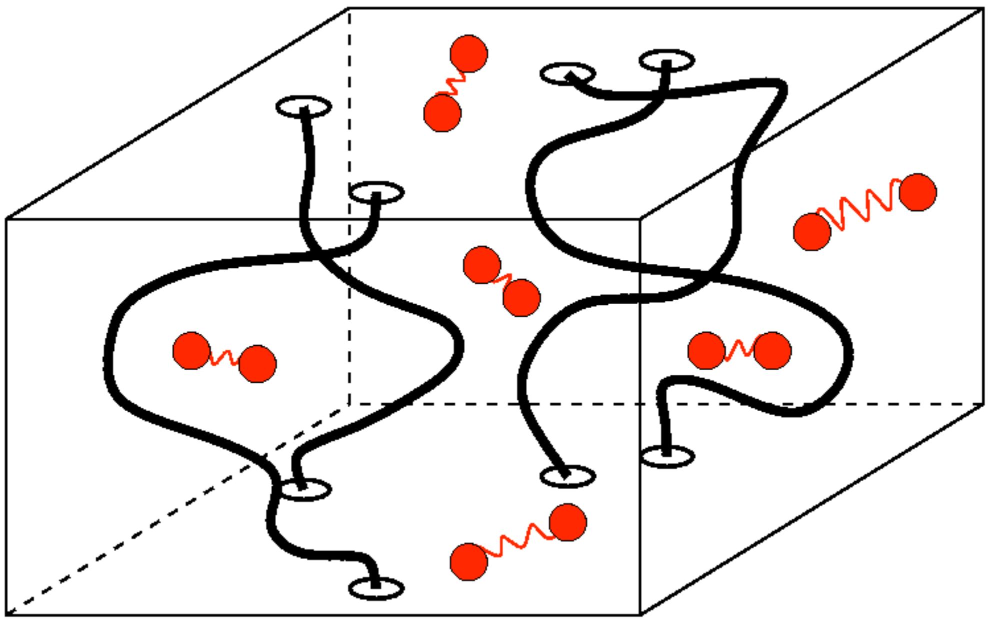
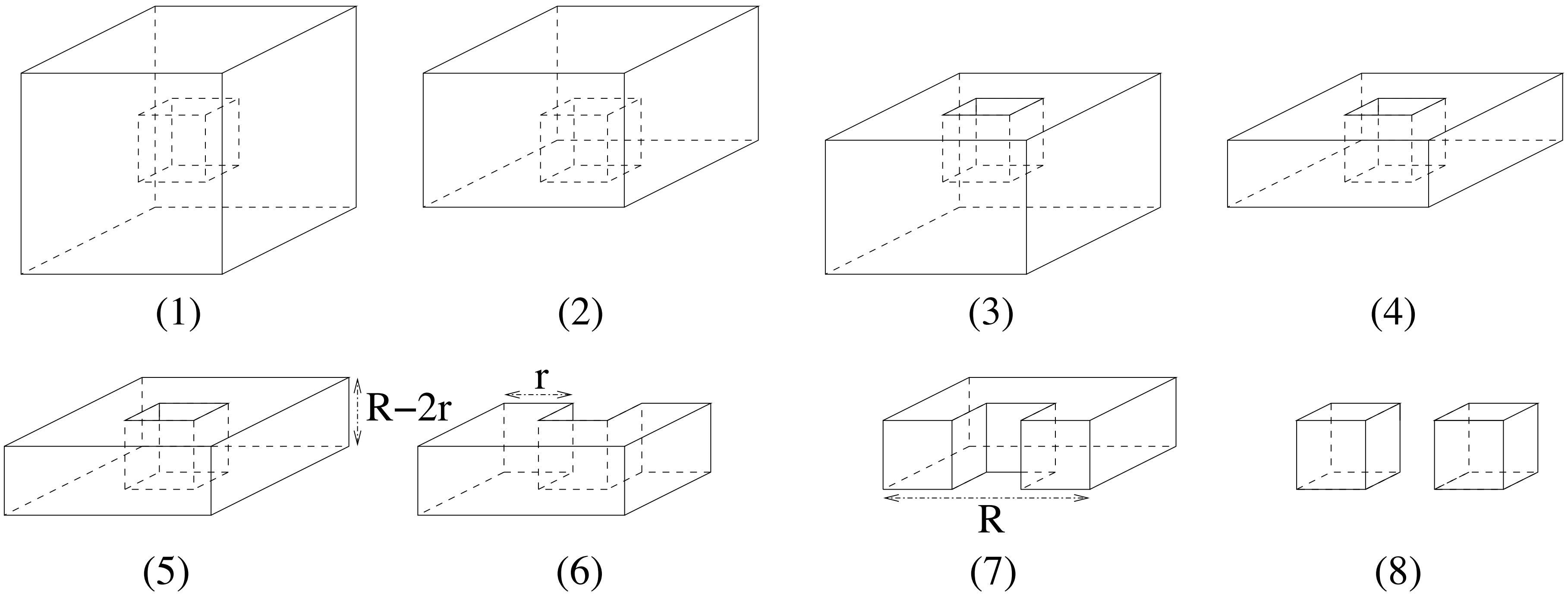


ROBUST!

$$\Gamma_1 = \langle \prod_{i\in\gamma_1} \sigma_i^z \rangle \to \frac{1}{N_{\gamma_1}} \sum_{\gamma_1} \langle \prod_{i\in\gamma_1} \sigma_i^z \rangle$$

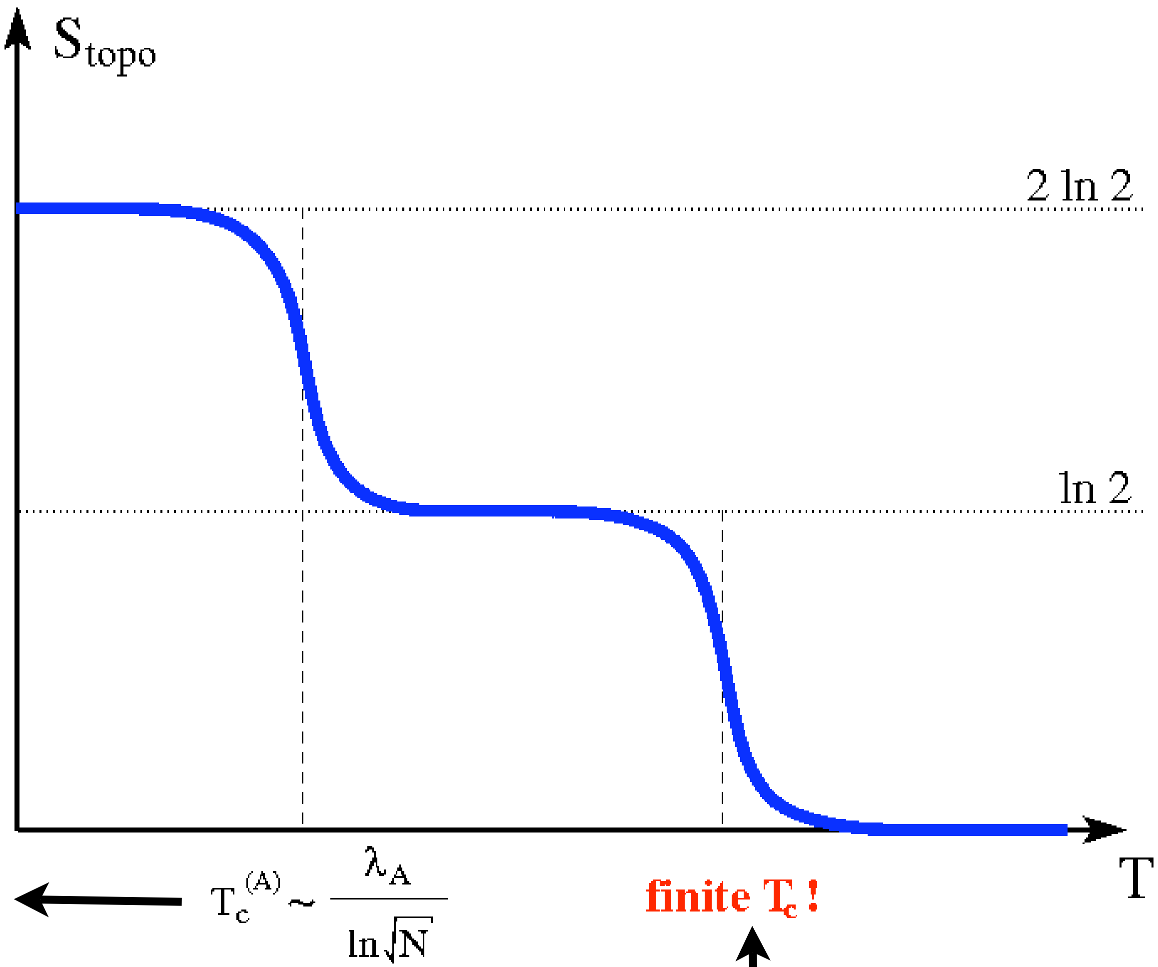
3D continued ...

- An exact expression can be derived for the von Neumann and topological entropies at any T
- Shell and donut type partitions, chosen symmetrically respective to $\mathcal{A}, \mathcal{B}$



$$S_{\text{topo}} = -S_{1A} + S_{2A} + S_{3A} - S_{4A} - S_{5A} + S_{6A} + S_{7A} - S_{8A}$$

Topological entropy in 3D



$$T_c = 1.313346(3)\lambda_B$$

- it is able to distinguish between the fragile and robust behavior
- the low temperature phase has non vanishing topological entropy
- the classical origin of each piece of the topological information is manifest (no need for hard constraints in 3D)

How one gets there...

$$
\begin{aligned}
\rho(T) &= \frac{1}{Z}\, e^{-\beta \hat{H}} \\
&= \frac{\sum_{\alpha,\beta} \langle \alpha | e^{-\beta\lambda_A S}\; e^{-\beta\lambda_B P} | \beta \rangle \; |\alpha\rangle\langle\beta|}{\sum_{\alpha} \langle \alpha | e^{-\beta\lambda_A S}\; e^{-\beta\lambda_B P} | \alpha \rangle}
\end{aligned}
$$

How one gets there...

$$\rho(T) = \frac{1}{Z}\, e^{-\beta \hat{H}}$$

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$$\sum_{\alpha} e^{\beta\lambda_A M_s(\alpha)} \langle \alpha | e^{\beta\lambda_B P} | \alpha \rangle$$

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$$\sum_{\alpha} e^{\beta\lambda_A M_s(\alpha)} \langle\alpha| e^{\beta\lambda_B P} |\alpha\rangle$$

$$\langle\alpha| \prod_p \left[\cosh\beta\lambda_B + \sinh\beta\lambda_B\, B_p\right] |\alpha\rangle$$

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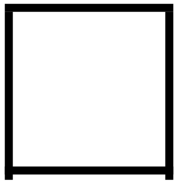

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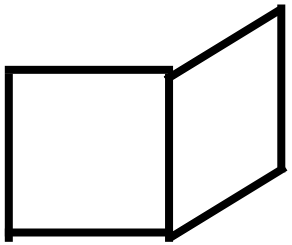

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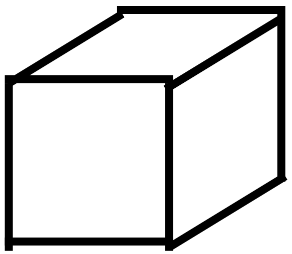

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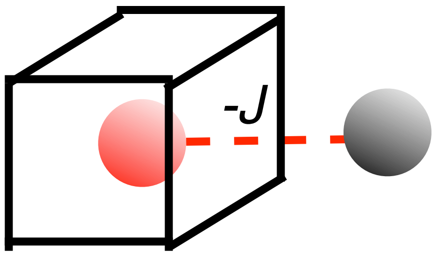

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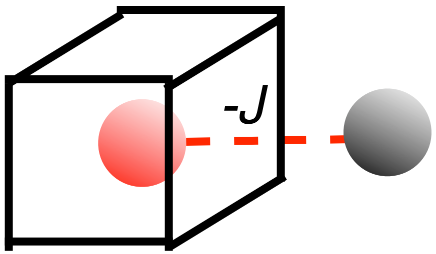

$$\frac{1}{2} \sum_{\mathcal{C}} \left[\cosh\beta\lambda_B\right]^{3N - N_{\mathrm{AF}}(\mathcal{C})} \left[\sinh\beta\lambda_B\right]^{N_{\mathrm{AF}}(\mathcal{C})}$$

$$= \frac{1}{2} (\sinh\beta\lambda_B \cosh\beta\lambda_B)^{3N/2} \sum_{\mathcal{C}} \exp\left(J \sum_{\langle i,j\rangle} S_i S_j \right)$$

How one gets there...

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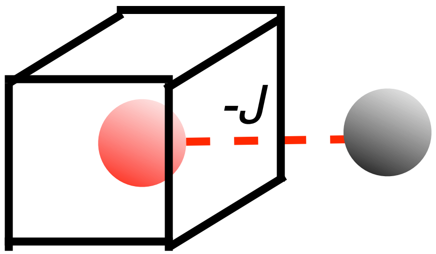

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3D Ising partition function with $e^{-2J} = \tanh \beta\lambda_B$

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$$\sum_{\alpha,\beta} e^{\beta\lambda_A M_s(\alpha)} \langle\alpha| e^{\beta\lambda_B P}|\beta\rangle\; |\alpha\rangle\langle\beta|$$

$$= \sum_{g\in G}\sum_{\alpha} e^{\beta\lambda_A M_s(\alpha)}\langle\alpha|e^{\beta\lambda_B P} g|\alpha\rangle\; |\alpha\rangle\langle\alpha|g$$

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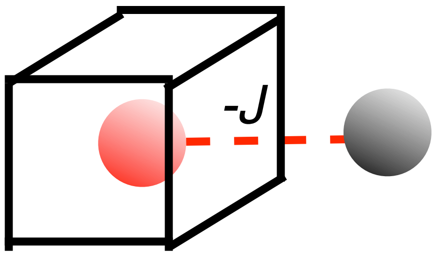

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3D Ising partition function with $e^{-2J} = \tanh\beta\lambda_B$

$$\rho(T) = \sum_{g\in G} \frac{Z_J^{\mathrm{tot}}(g)}{Z_J^{\mathrm{tot}}(1)} \sum_{\alpha} \frac{e^{\beta\lambda_A M_s(\alpha)}}{Z_s} |\alpha\rangle\langle\alpha|g$$

3D Ising partition function with $e^{-2J} = \tanh \beta\lambda_B$

3D random-bond Ising partition
(group element g translates into $-J$ bonds)

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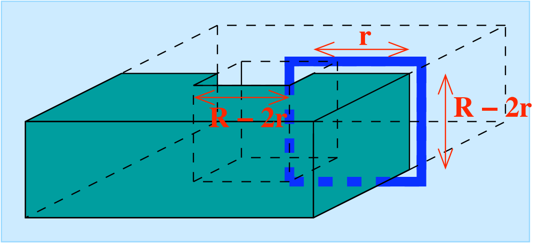
3D Ising partition function with $e^{-2J} = \tanh \beta\lambda_B$

plus collective ops.



$$S_{\text{topo}}(T) = S^{(S)}_{\text{topo}}(T/\lambda_A) + S^{(P)}_{\text{topo}}(T/\lambda_B).$$

$$S^{(S)}_{\text{topo}}(T/\lambda_A) = \begin{cases} \ln 2 & T = 0 \\ 0 & T > 0 \end{cases}$$

x-basis

$$S^{(P)}_{\text{topo}}(T/\lambda_B) = \begin{cases} \ln 2 & T < T_c \\ 0 & T > T_c \end{cases}$$

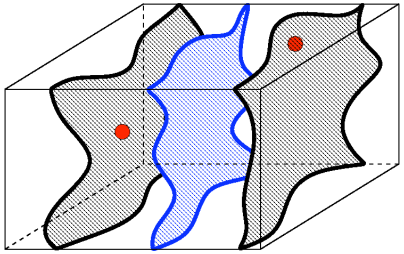
z-basis

$$S^{\text{3D}}_{\text{topo}}(T) = \begin{cases} 2\ln 2 & T = 0 \\ \ln 2 & 0 < T < T_c \\ 0 & T > T_c \end{cases}$$

Quantum topological order at T=0 follows from the superposition of the two classical pieces

What is the physical interpretation?

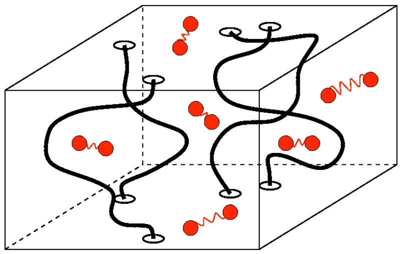
- Prepare the system (at T=0) in a superposition of different sectors at $t = t_i$
- Heat it to a temperature below T_c
- Cool it to T=0 at $t = t_f$



fragile

$$S^{(S)}_{\text{topo}}(T > 0) = 0$$

$$\langle \Psi_{\text{in}} | \prod_{i \in \xi} \sigma^x_i | \Psi_{\text{in}} \rangle \neq \langle \Psi_{\text{fi}} | \prod_{i \in \xi} \sigma^x_i | \Psi_{\text{fi}} \rangle$$



ROBUST!

$$S^{(P)}_{\text{topo}}(T < T_c) = \ln 2$$

$$\langle \Psi_{\text{in}} | \prod_{i \in \gamma} \sigma^z_i | \Psi_{\text{in}} \rangle = \langle \Psi_{\text{fi}} | \prod_{i \in \gamma} \sigma^z_i | \Psi_{\text{fi}} \rangle$$

Physical interpretation...

$$\langle \Psi_{\rm in} | \prod_{i\in\gamma} \sigma_i^z | \Psi_{\rm in} \rangle = \langle \Psi_{\rm fi} | \prod_{i\in\gamma} \sigma_i^z | \Psi_{\rm fi} \rangle$$

$$|\Psi_{\rm in}\rangle = \sum_{I=0}^{2^3-1} \sqrt{p_I}\, |I\rangle \quad \rightarrow \quad |\Psi_{\rm fi}\rangle = \sum_{I=0}^{2^3-1} \sqrt{p_I}\, e^{i\varphi_I}\, |I\rangle$$

PBIT - probability bit

What if ...

$$S^{(S)}_{\rm topo}(T<T_c)\neq 0 \qquad \text{and} \qquad S^{(P)}_{\rm topo}(T<T_c)\neq 0$$

$$\begin{cases} \langle\Psi_{\rm in}|\prod_{i\in\gamma}\sigma^z_i|\Psi_{\rm in}\rangle = \langle\Psi_{\rm fi}|\prod_{i\in\gamma}\sigma^z_i|\Psi_{\rm fi}\rangle \\ \langle\Psi_{\rm in}|\prod_{i\in\xi}\sigma^x_i|\Psi_{\rm in}\rangle = \langle\Psi_{\rm fi}|\prod_{i\in\xi}\sigma^x_i|\Psi_{\rm fi}\rangle \end{cases}$$

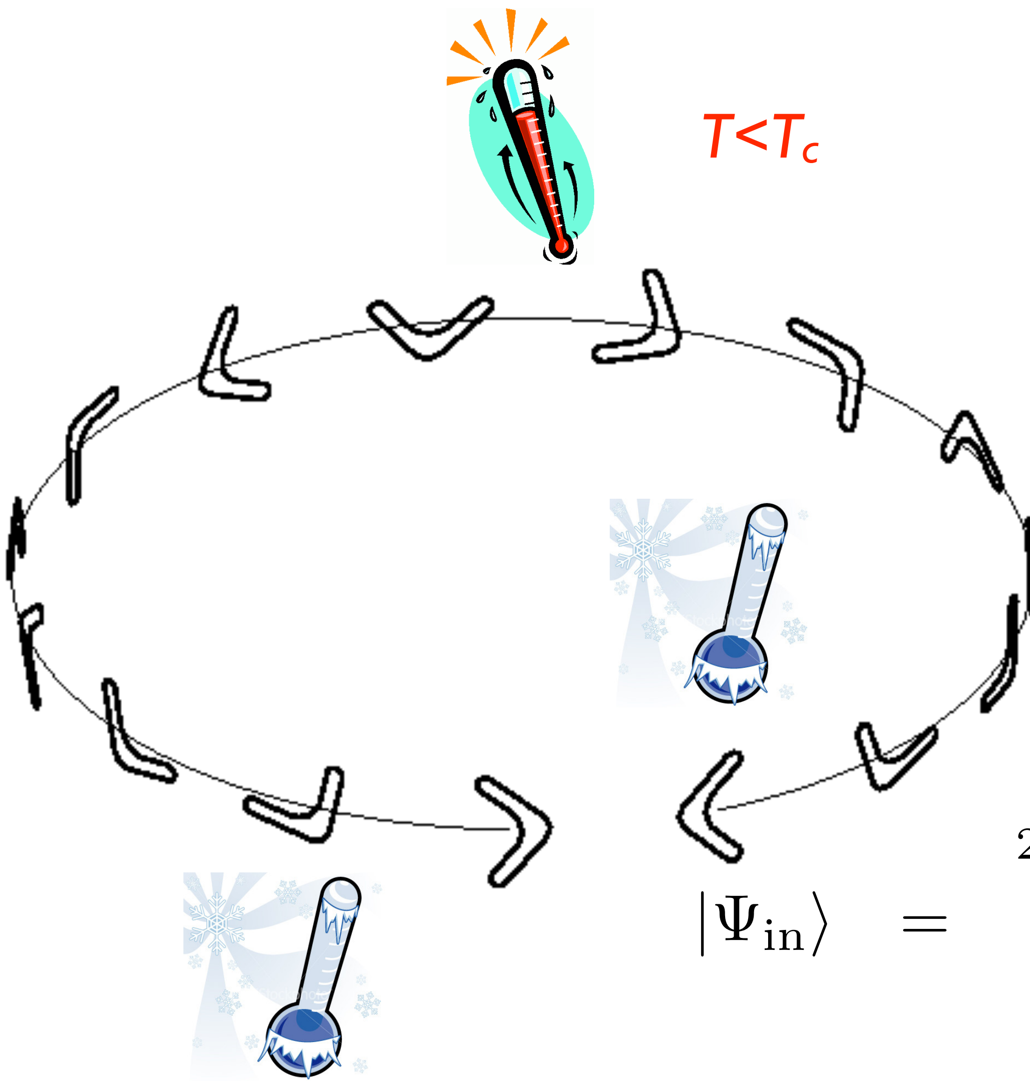
$$|\Psi_{\rm in}\rangle = \sum_{I=0}^{2^n-1}\sqrt{p_I}\,|I\rangle \quad \longrightarrow \quad |\Psi_{\rm fi}\rangle = e^{i\varphi}\sum_{I=0}^{2^n-1}\sqrt{p_I}\,|I\rangle$$

error correction schemes in 4D

Dennis, Kitaev, Landahl, and Preskill
J. Math. Phys. (2002).

<u>self-correcting system</u>

"Boomerang" effect



$$|\Psi_{\rm in}\rangle = \sum_{I=0}^{2^n-1} \sqrt{p_I}\,|I\rangle$$

$$|\Psi_{\rm fi}\rangle = e^{i\varphi} \sum_{I=0}^{2^n-1} \sqrt{p_I}\,|I\rangle$$

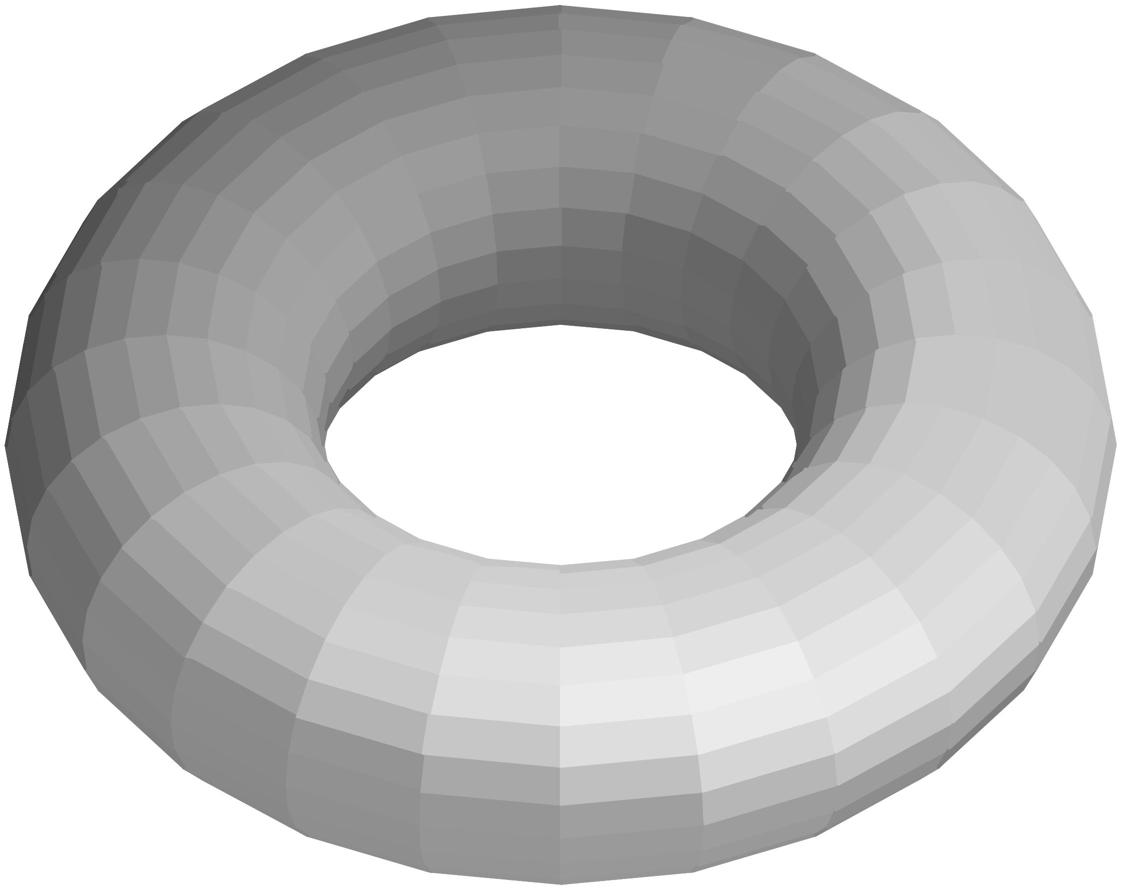
Toric-boson model

Hamma, Castelnovo & C[3]

$$H_{\rm Tb} = H_{\rm toric} + H_{\rm boson} + H_{\rm int}$$

$$H_{\rm boson} = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$$

$$H_{\rm int} = g_A \sum_s \frac{1-A_s}{2}\left(a_{\mathbf{x}_s} + a^\dagger_{\mathbf{x}_s}\right) + g_B \sum_p \frac{1-B_p}{2}\left(a_{\mathbf{x}_p} + a^\dagger_{\mathbf{x}_p}\right)$$

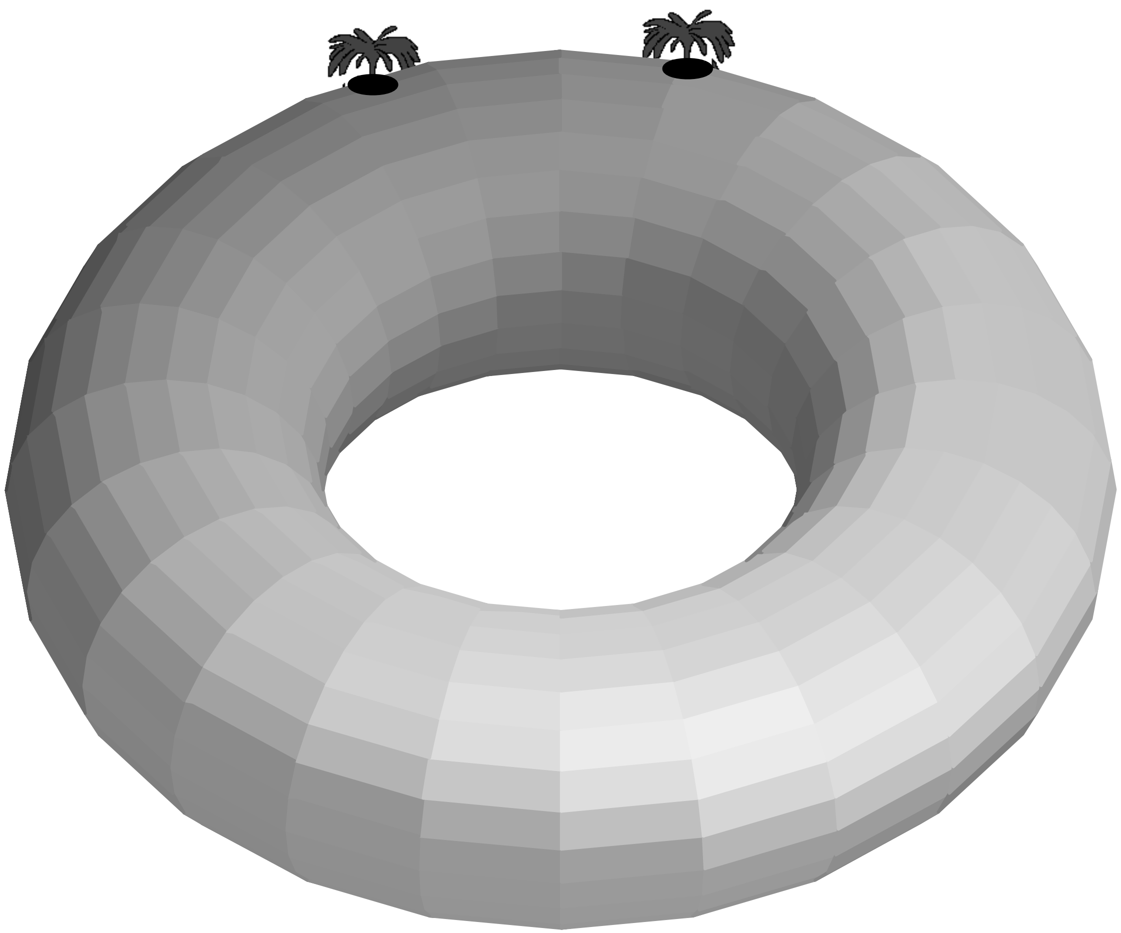


$$V_d\left(\{\mathbf{x}_1,\ldots,\mathbf{x}_{2N}\}\right) = \begin{cases} \frac{1}{m}\frac{2\Gamma(d/2)}{\pi^{d/2}}\frac{1}{2-d}\sum_{i<j}\; g_i g_j\; |\mathbf{x}_i - \mathbf{x}_j|^{2-d} & d \neq 2 \\ \frac{1}{m}\frac{2}{\pi}\sum_{i<j}\; g_i g_j\; \log|\mathbf{x}_i - \mathbf{x}_j| & d = 2, \end{cases}$$

$$G = \frac{1}{m}\frac{2\Gamma(d/2)}{\pi^{d/2}}$$

Toric-boson model

Hamma, Castelnovo & C[3]



$$H_{\text{Tb}} = H_{\text{toric}} + H_{\text{boson}} + H_{\text{int}}$$

$$H_{\text{boson}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \, a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}$$

$$H_{\text{int}} = g_A \sum_s \frac{1 - A_s}{2} \left(a_{\mathbf{x}_s} + a^{\dagger}_{\mathbf{x}_s}\right) + g_B \sum_p \frac{1 - B_p}{2} \left(a_{\mathbf{x}_p} + a^{\dagger}_{\mathbf{x}_p}\right)$$

$$V_d\left(\{\mathbf{x}_1, \ldots, \mathbf{x}_{2N}\}\right) = \begin{cases} \frac{1}{m} \frac{2\Gamma(d/2)}{\pi^{d/2}} \frac{1}{2-d} \sum_{i<j} \; g_i g_j \; |\mathbf{x}_i - \mathbf{x}_j|^{2-d} & d \neq 2 \\ \\ \frac{1}{m} \frac{2}{\pi} \sum_{i<j} \; g_i g_j \; \log|\mathbf{x}_i - \mathbf{x}_j| & d = 2, \end{cases}$$

$$G = \frac{1}{m} \frac{2\Gamma(d/2)}{\pi^{d/2}}$$

Conclusions

- Topological order can be well defined for $T \neq 0$, mixed states.
- There are well-defined classical counterparts in hard constrained models in 2D, or in 3D without need for hard constraints.
- Quantum order can (in some instances) be thought of superimposed classical topologically ordered system.
- Confined defects retain order at finite temperature.
- Topological entropy captures finite temperature order with equilibrium density matrices as starting point.
- Stable memories possible as long as system remains within the finite T topological phase.