Unified Description of (Some) Unitary and Nonunitary FQH States

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UIUC, October, 2008

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Single Component Fractional Quantum Hall States

- Unified description of FQH ground states and excitations (Read-Rezayi, Jain, Jack and others) in terms of squeezed polynomials
- Generalized Pauli principles: exclusion statistics, clustering, counting of multiplets
- Quasihole (and quasi-electron) excitations of Read-Rezayi, Jack states; pinned and un-pinned
- Non-Abelian Hierarchy States
- Specific Heat, electron and quasi-hole propagators, a first principle study
- Connection to Conformal Field Theory/Nonunitarity

Free Boson Many Body Wavefunctions

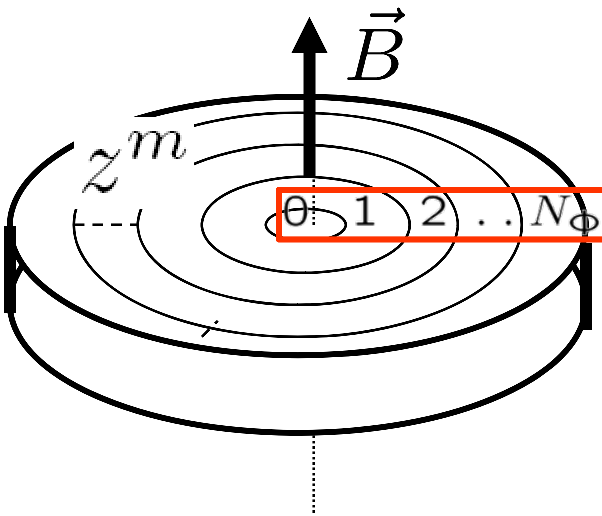
- Boson analog of Slater det. Orbital **occupation** basis

$$\begin{matrix} 0 & 1 & 2 & \ldots & N_\Phi \end{matrix}$$
$$[n_0, n_1, n_2 \ldots n_{N_\Phi}]$$

- Squeezing Rules in Orbital Space :
Rezayi and Haldane 1994

$$2002002002002002002 \quad \text{A}$$
$$2001102002002011002 \quad \text{B}$$

B Squeezed from A (A>B)



- Monomials (Permanents) = Det. with all signs positive

0 1 2 3 4 5 6 7 8...

$$[101010101\ldots] \longrightarrow (\ldots 8,6,4,2,0)$$

Orbital occupation ⟶ Monomial basis

$$m_\lambda(z_1,\ldots,z_N) = Per\left(z_i^{\lambda_j}\right) = Symm(z_1^{\lambda_1}\cdot\ldots\cdot z_N^{\lambda_N})$$

$$N=5 \quad [101010101] \rightarrow (8,6,4,2,0) \Longrightarrow m_{8,6,4,2,0} = Symm(z_1^8 z_2^6 z_3^4 z_4^2 z_5^0)$$

Squeezed Polynomials/Talk Outline

- Linear combinations of free many-body states squeezed from a root occupation

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} v_{\lambda\mu} m_\mu$$

- Once the root occupation is known, it is sometimes easy to obtain unique local Hamiltonian; always very easy to obtain part of the Hamiltonian

- With a bit of work, one can obtain rule for counting excitation multiplets - Pauli Principle: for Read-Rezayi – obtained by Haldane in 2006 - but rule much more general. From Pauli principle -> quantum dimension, effective central charge

- If (assumption) polynomial is CFT then very easy to obtain central charge, CFT scaling dimensions, identify unitary VS non-unitary

- Pinned quasihole wavefunctions easily formulated; same for quasielectron (of Read-Rezayi, for ex); can form abelian/non-abelian hierarchy

- Occupation number – like picture for quasihole: good chance of obtaining edge propagators/check some screening properties from first principles

Jack Polynomials (Jacks) $J^{\alpha}_{\lambda}(z_1,\ldots,z_N)$

Henry Jack, 1976

- Eigenstates of the Laplace Beltrami Operator are explicitly known

$$\mathcal{H}_{LB}(\alpha)J^{\alpha}_{\lambda}(z_1 \ldots z_N) = \epsilon_{\lambda}J^{\alpha}_{\lambda}(z_1 \ldots z_N)$$

α = Jack polynomial parameter

λ = monomial root occupation (partition) $=(\lambda_1, \lambda_2, \ldots, \lambda_N)$

- Decomposition of Jack polynomials in free boson many-body states known

$$J^{\alpha}_{\lambda}(z_1,\ldots,z_N) = m_{\lambda}(z_1,\ldots,z_N) + \sum_{\mu<\lambda} v_{\mu\lambda}(\alpha)m_{\mu}(z_1,\ldots,z_N)$$

μ squeezed from λ

Coefficients $v_{\mu\lambda}(\alpha)$ are known explicitly

- Jacks at $\alpha > 0$: 1D integrable at RG fixed point. Haldane Shastry, CS eigenst.

- Jacks at $\alpha < 0$ first studied in 2001! (Feigin et al math.QA/0112127 showed that the Jacks **span** the ideal of polynomials vanishing when k+1 arguments come together)

Laughlin and Moore-Read FQH States

• Annihilation operators on the Laughlin state $\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r$

$$D_i^{L,r} = \frac{\partial}{\partial z_i} - r \sum_{j(\neq i)}{}' \frac{1}{z_i - z_j}; \quad D_i^{L,r} \psi_L = 0$$

• Linear combination of the annihilation operators = Laplace Beltrami Operator

$$\sum_{i=1}^{N} z_i D_i^{L,1} z_i D_i^{L,r} = H_{LB}(\alpha_{1,r})$$

$$\alpha_{k,r} = -\tfrac{k+1}{r-1}$$

$$\mathcal{H}_{LB}(\alpha) = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j}^{N} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$$

• Annihilation operators on the Pfaffian state $D_i^{MR}\, Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j) = 0$

• Laughlin and Moore-Read (also Read-Rezayi): eigenstates of the Laplace-Beltrami; Large number of single component CFT FQH states are eigenstates of the same two-body operator.

How the Jacks Look

$$J^{-2}_{10101}$$

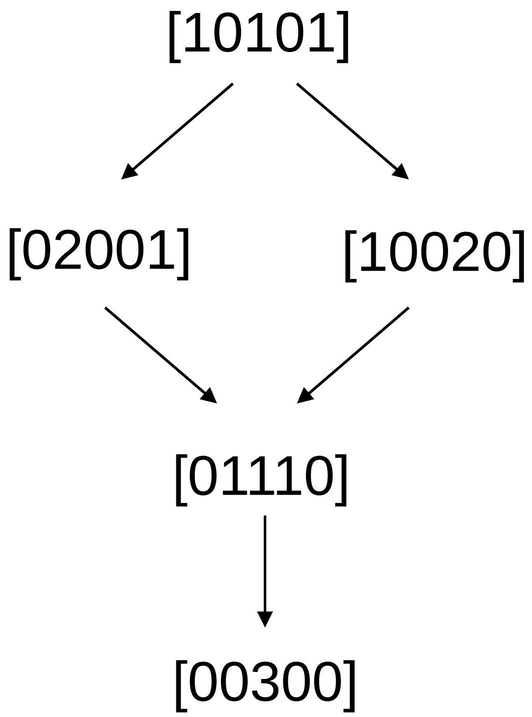
$$m_{4,\,2} - 2\,m_{3,\,3} - 2\,m_{4,\,1,\,1} + 2\,m_{3,\,2,\,1} - 6\,m_{2,\,2,\,2}$$

=

$$z_3^4 z_2^2 - 2\,z_3^4 z_2 z_1 + z_3^4 z_1^2 - 2\,z_3^3 z_2^3 + 2\,z_3^3 z_2^2 z_1 + 2\,z_3^3 z_2 z_1^2$$
$$- 2\,z_3^3 z_1^3 + z_3^2 z_2^4 + 2\,z_3^2 z_2^3 z_1 - 6\,z_3^2 z_2^2 z_1^2 + 2\,z_3^2 z_2 z_1^3 + z_3^2 z_1^4$$
$$- 2\,z_3 z_2^4 z_1 + 2\,z_3 z_2^3 z_1^2 + 2\,z_3 z_2^2 z_1^3 - 2\,z_3 z_2 z_1^4 + z_2^4 z_1^2 - 2\,z_2^3 z_1^3 + z_2^2 z_1^4$$

=

$$(z_1 - z_3)^2\,(z_2 - z_3)^2\,(-z_2 + z_1)^2$$



Generalized Pauli Principle

- Model WF: **Highest Weight** (no quasiholes) and **Lowest Weight** (no quasiparticles)

  - These **uniquely** define ALL good FQH Jacks :

$$\nu = \frac{k}{r}: \ |[k0^{r-1}k0^{r-1}k0^{r-1}...]\rangle \rightarrow J_{n^{v}(k,r)}^{-\frac{k+1}{r-1}}(z_1, ..., z_N)$$

$$n^{v}(k,r) = k0^{r-1}k0^{r-1}k0^{r-1}...0^{r-1}k$$

  - r=2 is the Read-Rezayi Z_k sequence. Laughlin(k=1), Read-Moore(k=2)

The FQH ground states above are the maximum density states satisfying a **generalized Pauli principle of not more than k particles in r consecutive orbitals!**

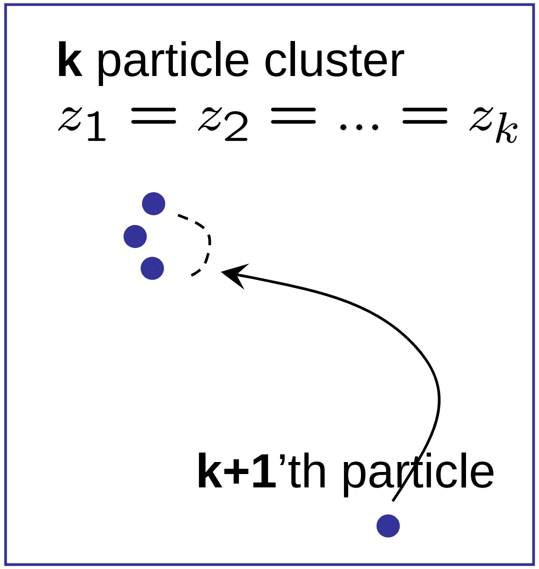
  - For r=2, see Haldane, APS 2006
  - Natural generalization of Read-Rezayi states; q-hole excit also Jacks but NOT qp

  - Torus GS degenaracy: $N_{GS} = \frac{(k+r-1)!}{k!(r-1)!}$

Clustering Conditions

• Pauli principle + squeezing rule = Clustering = (part of) Hamiltonian. Take GS:

$$[k0^{r-1}k0^{r-1}k0^{r-1}...k0^{r-1}k]$$



•Form a **k** particle cluster at origin:

$$[00^r[0^r k0^{r-1}k0^{r-1}...k0^{r-1}k]$$

$$\equiv \prod_{i=k+1}^{N} z_i^r \cdot [k0^{r-1}k0^{r-1}...k0^{r-1}k]$$

• For the **(k,r)** sequence, the GS and quasihole Jack WF vanish as the **r**'th power of the difference between coordinates:

$$J_{n(k,r)}^{-\frac{k+1}{r-1}}(z_1 = ... = z_k, z_{k+1}, ..., z_N) \sim \prod_{i=k+1}^{N} (z_1 - z_i)^r$$

• Zero modes of pseudopotential Hamiltonians:

$$\sum_{i=0}^{r-1} V_{k+1}^i$$

• For r>3 Jack not unique GS of these psudopot Hamilt

Central Charge and Electron Scaling Dimension

- Is a coefficient embedded in the polynomial **ground-state** wave-function

$$\psi_{a_1}(z_1)\psi_{k-a_1}(z_2) = \frac{C_{a_1,k-a_1}}{(z_1-z_2)^{h_{a_1}+h_{k-a_1}}}\left(1+\frac{2h_{a_1}}{c}(z_1-z_2)^2 T(z_1)\right)$$

- Put k-1 particles at 1 point (say origin) to form conjugate ψ_{k-1} field

$$\frac{J_{10^{r-1}k0^{r-1}k\ldots k0^{r-1}k}(z_k,z_{k+1},\ldots,z_N)}{\prod_{i=k}^{N} z_i^{\frac{k-1}{\nu}} \prod_{i<j=k}^{N}(z_i-z_j)^{\frac{1}{\nu}}}$$

$$\boxed{2h_{el} = \frac{k-1}{\nu} = \frac{r(k-1)}{k}}$$

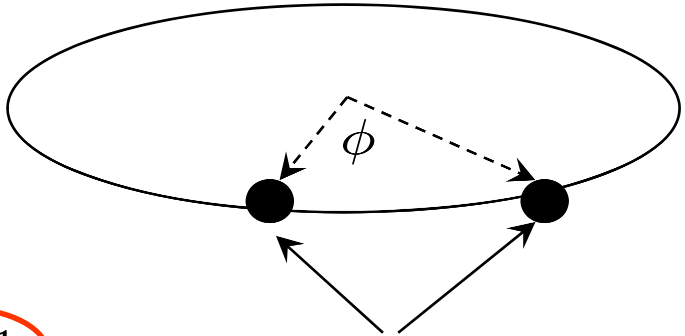
- Expand over z_k, isolate quadratic term

$$\boxed{c = (k-1)\left(1-\frac{k(r-1)^2}{k+r}\right)}$$

Electron Scaling Dimension Without CFT Assumption

- For Laughlin States (Wen,1990):

$$G(\phi) \sim \frac{1}{\left(\sin\left(\frac{\phi}{2}\right)\right)^{g_e}}$$



$$n_M = \int_0^{2\pi} e^{-iM\phi} G(\phi) \sim (N_\phi - M)^{g_e - 1}$$

Luttinger Liquid Behavior

$$n_M \sim \frac{(N_\phi - M + g_e - 1)!}{(g_e - 1)!(N_\phi - M)!}$$

$$g_e = r$$

g_e	n_{N_ϕ}	$n_{N_\phi - 1}$	$n_{N_\phi - 2}$	$n_{N_\phi - 3}$	$n_{N_\phi - 4}$	$n_{N_\phi - 5}$
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126

k	r	n_{N_ϕ}	$n_{N_\phi - 1}$	$n_{N_\phi - 2}$
1	2	1	2	2.88
2	2	1	2.21	3.4
4	3	1	3.07	6.24
3	4	1	3.93	8.43

Unpinned Excitation Wavefunctions

- Maintain Pauli principle of (k,r) statistics (not more than k particles in r consecutive orbitals) but add fluxes (zeroes) on the sphere:

Laughlin GS (k,r)=(1,2) $|[10101]\rangle$

1-Quasihole Multiplet L=N/2

$|[010101]\rangle$

$|[100101]\rangle$

$|[101001]\rangle$

$|[101010]\rangle$

Pfaffian GS (k,r)=(2,2) $|[20202]\rangle$

Abelian Quasiholes

$|[020202]\rangle$

$|[200202]\rangle$

$|[202002]\rangle$

$|[202020]\rangle$

Non-Abelian Fractionalized Quasihole

$|[110202]\rangle$

$|[111102]\rangle$

$|[111111]\rangle$

$|[201111]\rangle$

$|[202011]\rangle$

$|[201102]\rangle$

- For r=2 (Read-Rezayi sequence) this gives the known counting of CFT (Haldane 2006)

- For any (k,r) have all counting of modes (have analytic formulas)

Edge Thermal Hall Coefficient

- Edge entropy of non-abelian k/r states: We performed High Temperature expansion

Fermi Sea

Excitations

$$\ldots k0^{r-1}k0^{r-1}\ldots k0^{r-1}k0^{r-1}k\,|\,00000\ldots00\ldots$$

$2\pi/L$

$$Z = Tr\left(\exp(-\beta H)\right) = \sum_{D=1}^{\infty} N_D q^D$$

$$q = e^{-\frac{2\pi\beta v_F}{L}} \qquad F = -T\ln(Z)$$

- We computed N_D using the theory of partitions (Andrews book)

$$-T\frac{\partial^2 F}{\partial T^2} = \frac{\pi L T}{3 v_F} c$$

- c =(effective) central charge in CFT; (asymptotic growth)

effective central charge of $W_k(k+1, k+r)$

$$c_{\text{eff}} = 1 + \frac{r(k-1)}{r+k}$$

U(1) charge part

Non-abelian part >0

Pinned Quasiholes

- Coherent State superposition of un-pinned quasiholes (Jack polynomials)

$$\prod_i^N (z_i - z_A) \prod_{i<j}^N (z_i - z_j)^r = \sum_{i=1}^N z_A^i J_i(z_1, ..., z_N)$$

- k-1 fractionalized quasiholes at the origin, one at z_A. Example for k=2 and k=3:

$$|0\rangle \rightarrow |0202...0202\rangle$$

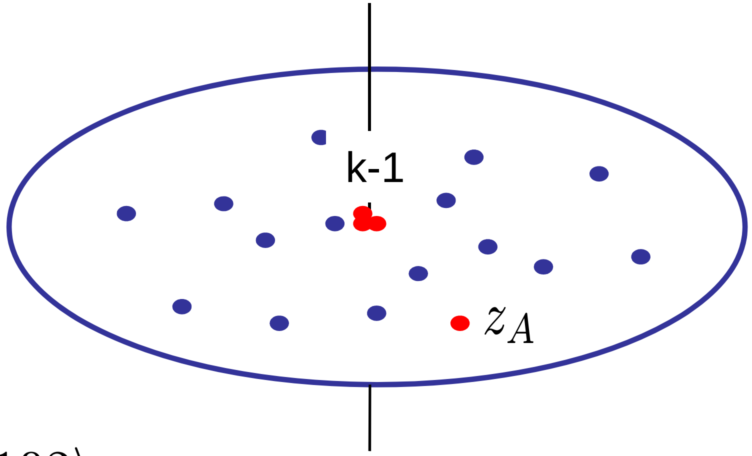
$$|1\rangle \rightarrow |1102...0202\rangle$$

$$|2\rangle \rightarrow |1111...0202\rangle$$

$$\vdots$$

$$|\tfrac{N}{2} - 1\rangle \rightarrow |1111...1102\rangle$$

$$|\tfrac{N}{2}\rangle \rightarrow |1111...1111\rangle$$



$$|0\rangle \rightarrow |0303...0303\rangle$$

$$|1\rangle \rightarrow |1203...0303\rangle$$

$$|2\rangle \rightarrow |1212...0303\rangle$$

$$\vdots$$

$$|\tfrac{N}{3} - 1\rangle \rightarrow |1212...1203\rangle$$

$$|\tfrac{N}{3}\rangle \rightarrow |1212...1212\rangle$$

$$\Psi(z_A, 0^{k-1}; z_1, ..., z_N) = \sum_{i=0}^{\frac{N}{k}} \frac{1}{k^i} z_A^i |i\rangle$$

Quasihole Scaling Dimension

Also with S. Simon

- Bring the remaining quasihole at z_A close to the k-1 existing at the origin

- Even if we assume a CFT, since quasihole wavefunction is (unnormalized) polynomial, obtaining scaling dim is not trivial (unlike in the electron case)

$$\Psi(z_A, 0^{k-1}; z_1, ..., z_N) = \sum_{i=0}^{\frac{N}{k}} \frac{1}{k^i} z_A^i |i\rangle$$

- *Assume* CFT, then using Jack properties we compute the quasihole scaling dim

$$z_1 \to z_A \to 0$$

$$\langle \underbrace{\sigma(0)...\sigma(0)}_{k-1} \sigma(z_A)\psi(z_1)...\psi(z_N)\rangle = f(z_A, 0)\left(\psi_{GS} + \frac{z_A^2}{z_i^2}\left(\frac{1}{2}\frac{1}{k}(1-\frac{1}{k})\psi_{GS}\right)\right)$$

$$\boxed{h_\sigma = \frac{c}{2kr}}$$

$$\to \quad 2\frac{h_\sigma h_\psi}{c}$$

- In these models, negative scaling dimension related to negative central charge
- Qh Scaling dimension = coefficient embedded in the qh wavefunction

CFT Connection

$$c = (k-1)\left(1 - \frac{k(r-1)^2}{k+r}\right)$$ $$c_{\text{eff}} = 1 + \frac{r(k-1)}{r+k}$$ $$h_{el} = \frac{r(k-1)}{2k}$$ $$h_\sigma = \frac{c}{2kr}$$

- c (as coefficient in wavefunction) = c_{eff} (as from counting edge **excitations)** only for r=2; for r>2 (and k>1), c < 0, hence non-unitary CFT

- electron scaling dim. same from wavefunction and occupation number numerics

- (k,r) Jacks are corr. func of $W_k(k+1, k+r)$ models (conjectured: Jimbo, et al,2003; strong evidence this is true: BAB, Haldane 2008, Fusion rules: Ardonne 2008).

- r>2 (and k>1), c < 0, non-unitary CFTs;

- Conjecture (Read, 2008): Non-Abelian sector doesn't screen for nonunitary CFT

- Hard to check from first principles. If nonunitary CFT = wavefunction: immediate consequence:

$$h_{qh} < 0 \rightarrow \sigma(z_1)\sigma(z_2) \sim (z_1 - z_2)^{2|h_{qh}|}$$

One Quasiparticle States (Abelian)

Quasiparticle States: $L^{+}\psi = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}\psi = 0$

Start with Laughlin state: $|1010101...101\rangle$

Add 3 fluxes: $|0001010101...101\rangle$

Add 2 particles at north pole: $|2001010101...101\rangle$

Generalized Clustering properties satisfied by polynomials:

$P(z_1, z_1, z_1, z_4, z_5, ...) = 0$ as $l = 3$ $\quad P(z_1, z_1, z_3, z_3, z_5, z_6 ...) = 0$

Not Jacks but generalized Jacks (BAB and Haldane, PRB 2007; arxiv 2008)

One Quasiparticle States (Abelian)

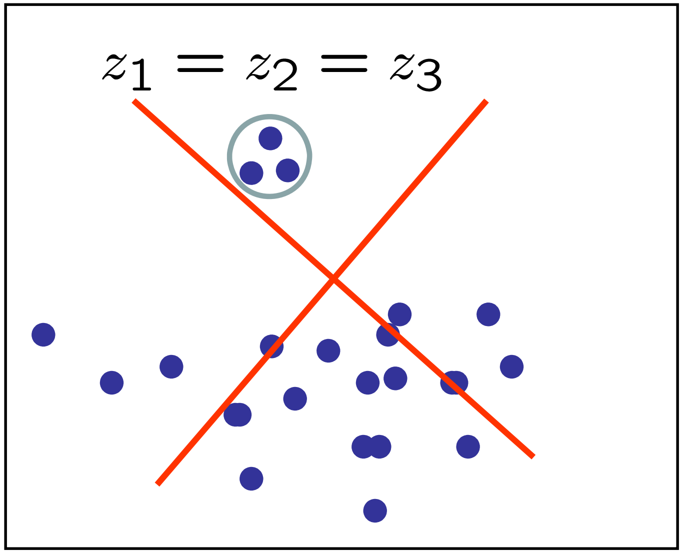
• Laughlin quasiparticle satisfies first clustering but not second

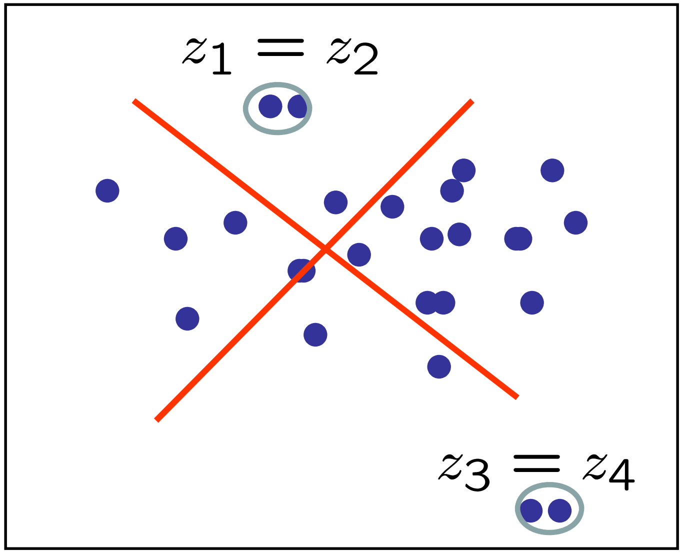
$$\psi_L = \prod_{i=1}^{N} \frac{\partial}{\partial z_i} \prod_{i<j}^{N} (z_i - z_j)^r;$$

• Our quasiparticle has more zeroes, due to generalized clustering

$$P^{1\ qp}_{Jain} = \begin{pmatrix} z_1^\star & z_2^\star & \dots & z_N^\star \\ 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^{N-2} & z_2^{N-2} & \dots & z_N^{N-2} \end{pmatrix}$$

$$P^{1qp}_{Jain} \prod_{i<j}^{N} (z_i - z_j)$$





Jack Hierarchy States

Bosonic state at $\nu = \frac{2}{3}$ (fermionic $\nu = \frac{2}{5}$) by dumping $\frac{N}{2}$ quasiparticles in Laughlin $\nu = \frac{1}{2}$ (fermionic $\nu = \frac{1}{3}$) state.

	Jack Quasiparticle	Jain Quasiparticle
1	$\|20010101010101\rangle$	$\|20010101010101\rangle$
2	$\|2002001010101\rangle$	$\|2010101010101\rangle$
3	$\|200200200101\rangle$	$\|201011010101\rangle$
4	$\|2002002002\rangle$	$\|2010110102\rangle$

Hierarchy leads to the Jack polynomial state: (k,r)=(2,3)

This is identical to Gaffnian state, previously proposed by **Simon, Rezayi, Cooper 2007**; Non-Unitary.

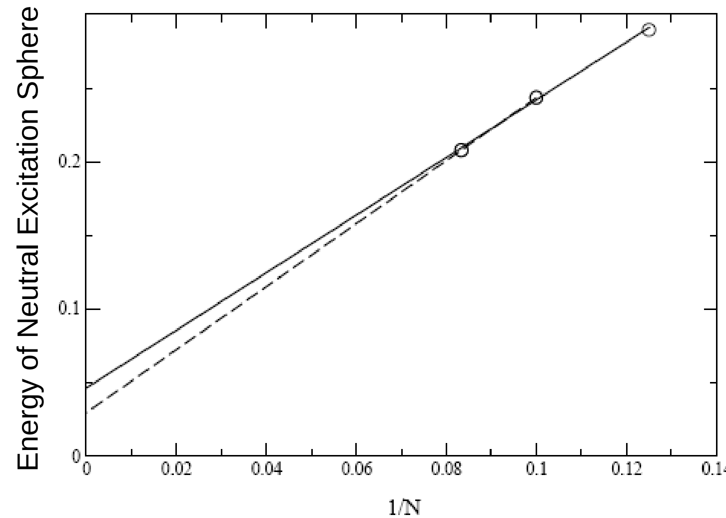
Numerics on the Gaffnian State

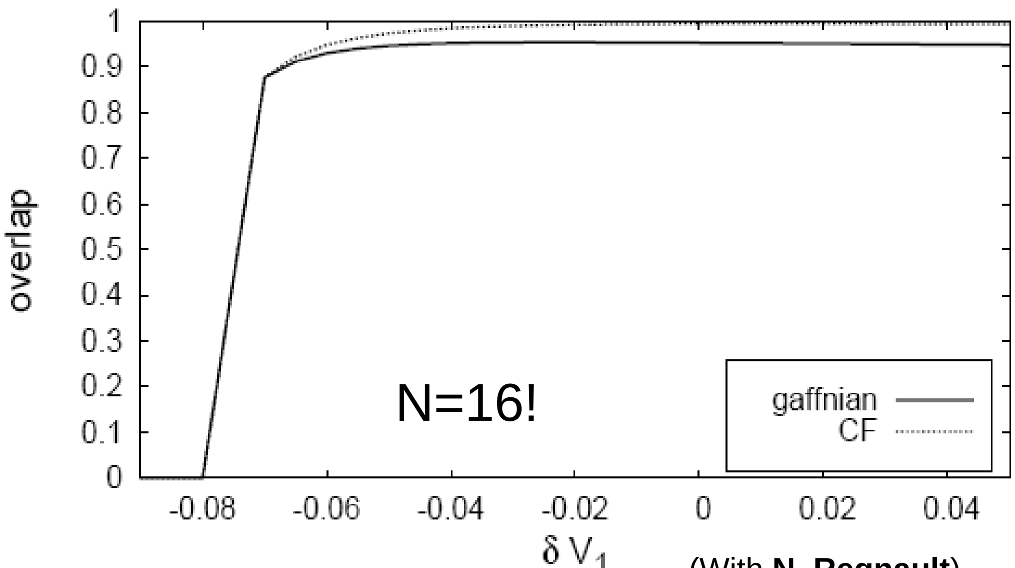
- Overlaps >0.96 on sphere (Rezayi; Regnault) for N=12, 14

N	6	8	10	12	14
$\mathcal{O}_{\nu=2/5}$	0.991 (3)	0.982 (4)	0.977 (5)	0.972 (5)	0.968 (6)
N	6	9	12	15	
$\mathcal{O}_{\nu=3/7}$	0.993 (3)	0.979 (4)	0.963 (5)	0.954 (7)	

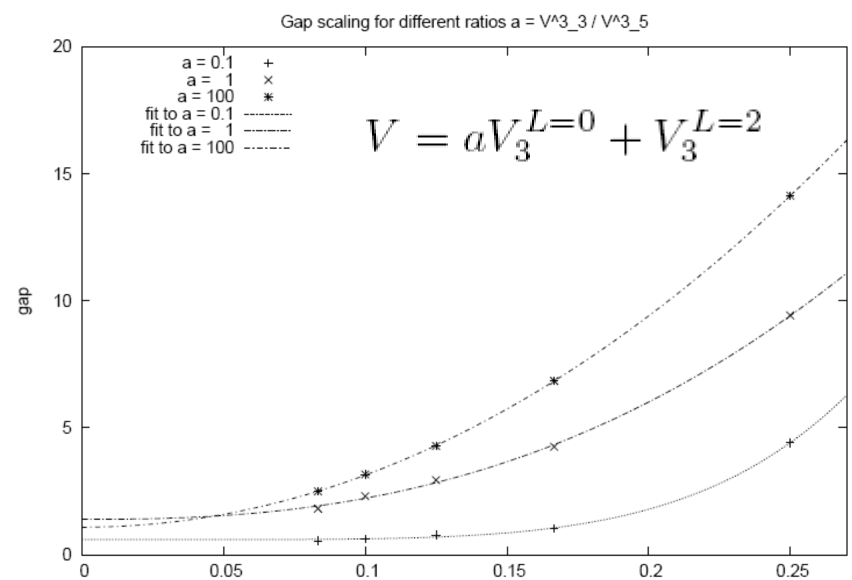
N. Regnault[1], M. O. Goerbig[2] and Th. Jolicoeur[3]

Simon, Rezayi, Cooper, Berdnikov, PRB 2007





(With **N. Regnault**)

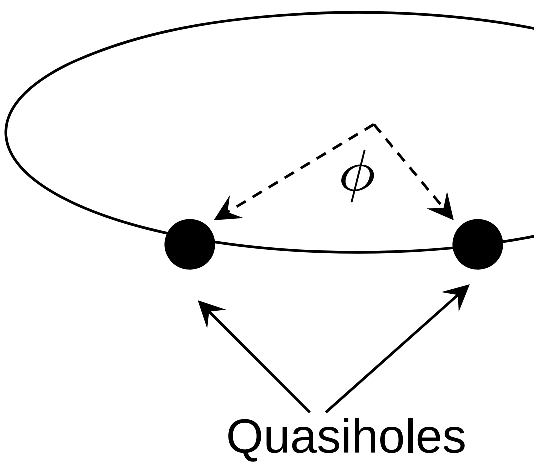


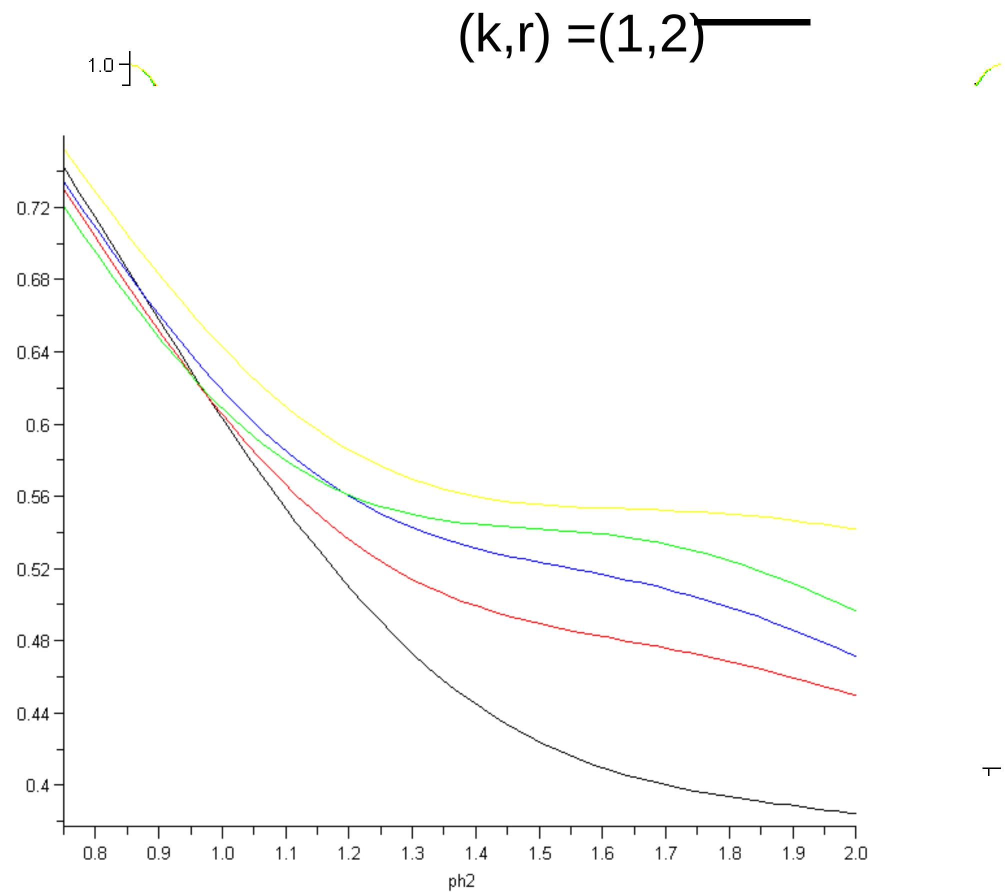
(With **Ronny Thomale** and M. Greiter)

Quasihole Propagators

- For Laughlin States:

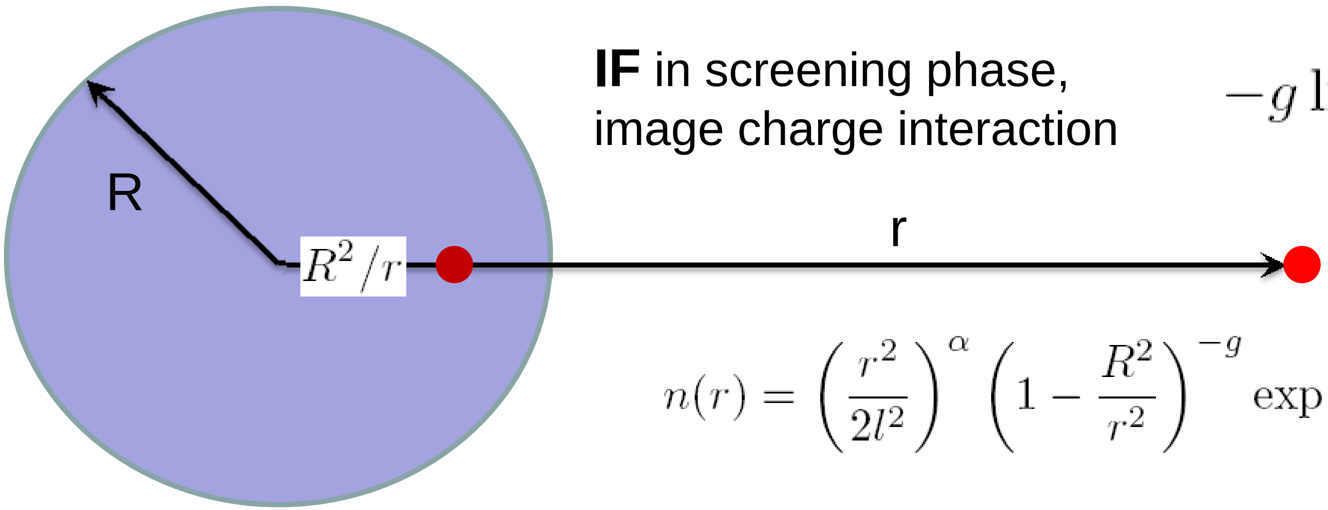
$$G(\phi) = \frac{1}{(\sin(\phi/2}$$





Gaffnian state (Jack (2,3)) has negative scaling dimension in nonabelian sector; if only the U(1) sector screens, then g=1/6 – 1/10

Quasihole "Occupation Number" and Plasma Screening



IF in screening phase, image charge interaction $-g \ln\left(r - R^2/r\right)$

$$n(r) = \left(\frac{r^2}{2l^2}\right)^{\alpha}\left(1 - \frac{R^2}{r^2}\right)^{-g}\exp\left(-\frac{qr^2}{2l^2}\right)$$

For (k,r) sequence quasiholes, can prove exactly $\alpha = \frac{N}{k},\ g = g_{qh},\ q = 1/r$

g cannot be obtained exactly (yet) but: We know the expression of n(r) in terms of Jacks: $|j\rangle$

$$n_{qh}(r)\exp(qr^2/2l^2) = \sum_{j=0}^{N/k}\left(\frac{r}{k}\right)^{2j}\langle j|j\rangle$$

k=2, r=2 Moore-Read / k=3, r=2 Read-Rezayi / Gaffnian

$$\left\langle \frac{N}{k} - i \middle| \frac{N}{k} - i \right\rangle = c\begin{pmatrix} g+i-1 \\ i \end{pmatrix}\left(\frac{R^{2i}}{k^{2i}}\right)$$

N=6: g=0.392555; N=8: g=0.379...; N=9: g=0.3...; N=10: g=0.378535; Conjecture: N=Large: g=0.33333

CFT prediction 3/8=0.375; 1/3 ... First first principle calculation

More on Zeroes of Jain States at k/k+1

•Divide a Jain state at k/2k+1 by one filled Landau level to mod out trivial fermionic term.
• All Jain states are squeezed polynomials:

2/3: $201011011011011011 0110...110110110110110102$

3/4: $3010110111011101110 1110...1110111011101110103$

4/5: $40101101110111101111 0...11110111101110110104$

• Jain state at k/k+1 zero mode of V_{k+1}^{0}

$$Jain^{\nu=\frac{k}{k+1}}(\underbrace{z_1,...,z_1}_{k},z_{k+1},z_{k+2},...,z_N)\sim\prod_{i=k+1}^{N}(z_1-z_i)^2$$

•Jain state NOT uniquely defined by the root partition (so no full Hamiltonian, just part)

• Method used for large size N=16 Jain entanglement entropy (**Rengault**, BAB)

Series of States With Exact Hamiltonians

With R. Thomale and M. Greiter

$$\nu = 1/2 \quad 101010101010101\ldots \quad H = V_2^0$$
$$\nu = 2/3 \quad 200200200200200 2\ldots \quad H = V_3^0 + V_3^2$$
$$\nu = 3/4 \quad 20102010201020102\ldots \quad H = V_3^0 + V_3^5$$
$$\nu = 4/5 \quad 2020020200202 00202\ldots \quad H = V_3^0 + V_3^8$$
$$\nu = 5/6 \quad 2020102020102020 10\ldots \quad H = V_3^0 + V_3^{13}$$
$$\nu = 6/7 \quad 2020200202020020202 00\ldots \quad H = V_3^0 + V_3^{18}$$

•States at k/k+1, different shift than Jain states

•Pauli Principle available for all these states; counting

• When k large, Moore-Read state. Same unique states exist for all Read Rezayi states

Conclusions

- A unified description of large class of FQH states in terms of squeezed polynomials (good for numerics, see Prodan and Haldane; Jain state up to 16 particles, see Regnault, BAB)

- Easy to identify part of the Hamiltonian, clustering conditions, degeneracy counting, quasihole multiplet counting, central charge, effective central charge, electron and quasihole scaling dimensions (as CFT would define them), pinned quasihole wavefunctions. All these ONLY use the polynomial structure and NOT the scalar product

- Scalar product used to compute electron propagators on the edge; seem to match CFT for both unitary and nonunitary

- Scalar product used to compute quasihole propagators on the edge: very good CFT match for unitary, no match for non-unitary (but overscreening of the charge sector).

- New series of hierarchy states w. unique Hamiltonians, ending in RR states