Chains of “Interacting” Non-Abelian Quasiparticles

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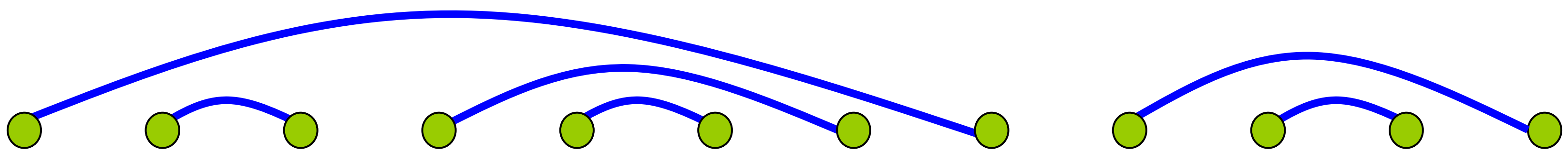
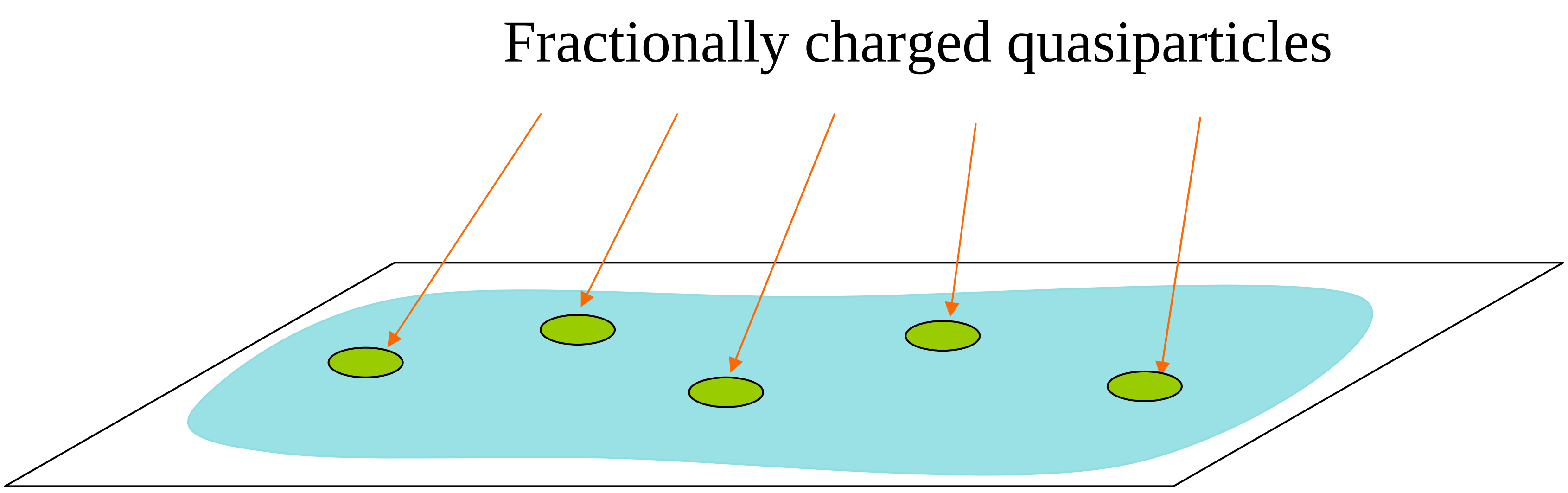


Work with: Huan Tran (FSU), Lukasz Fidkowski (Caltech), Kun Yang (FSU), Gil Refael (Caltech), Joel Moore (Berkeley).

NEB, K. Yang, Phys. Rev. Lett. 99, 140405 (2007).
L. Fidkowski, G. Refael, NEB, J. Moore arxiv: 0807.1123
H. Tran, NEB, in preparation.

Support: US DOE

Non-Abelian FQH States (Moore, Read '91)



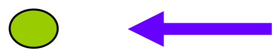
Essential features:

A degenerate Hilbert space whose dimensionality is **exponentially large in the number of quasiparticles**.

States in this space **can only be distinguished by global measurements** provided quasiparticles are far apart.

$SU(2)_k$ Non-Abelian Particles

1. Particles have topological charge **s** = 0, 1/2, 1, 3/2 , …, k/2

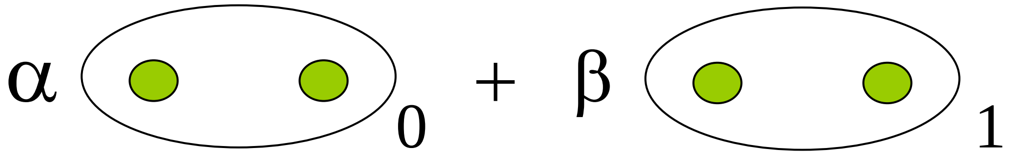

topological charge = ½

2. “Fusion Rule” for adding topological charge:

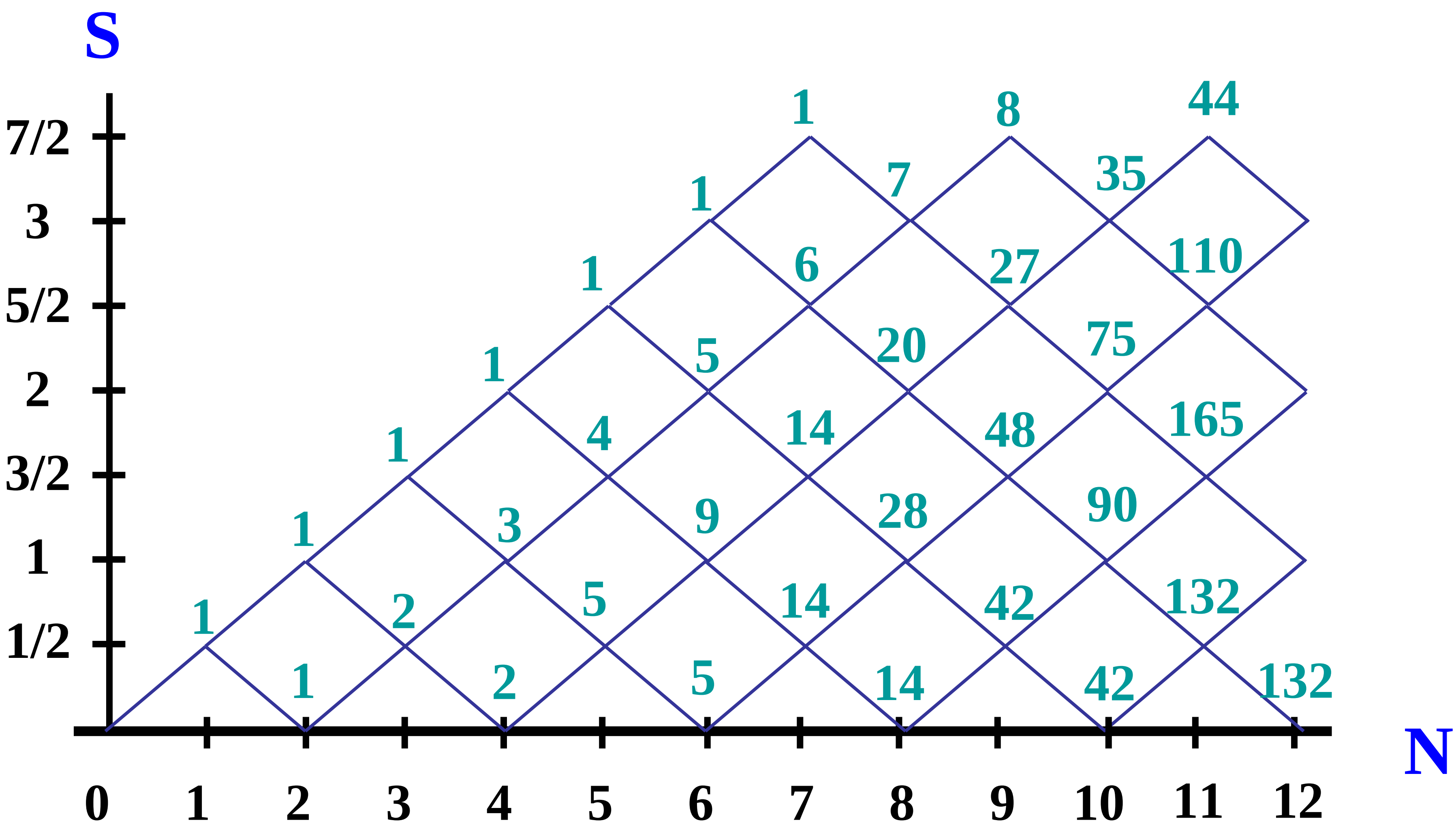
$$s_1 \otimes s_2 = |s_1 - s_2| \oplus \left(|s_1 - s_2| + 1 \right) \oplus \cdots \oplus \min\left[s_1 + s_2, k - (s_1 + s_2) \right]$$

For example: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

➔ Two ● particles can have total topological charge **0** or **1**.

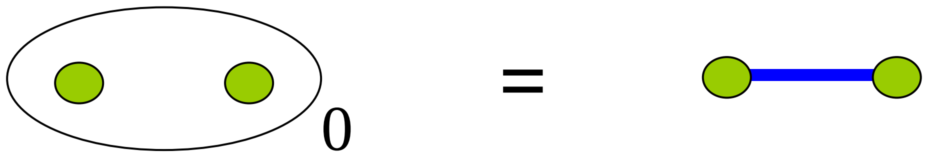


$k \to \infty$; **Ordinary Spin-1/2 Particles**

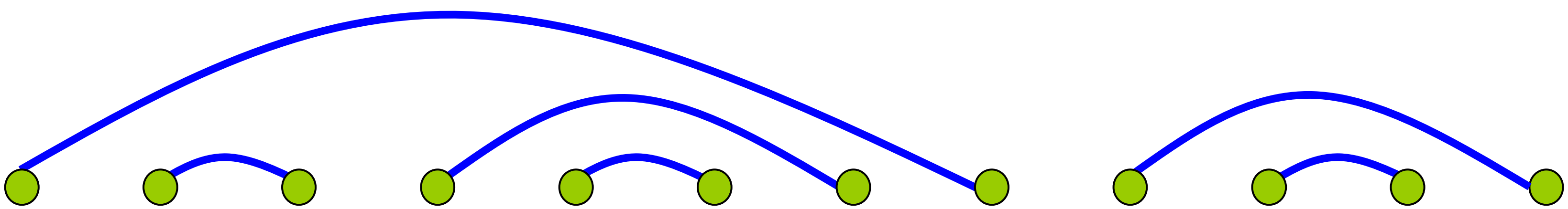


Dim(*N*) ~ 2^N

Valence Bonds Basis



Non-crossing valence bond basis:



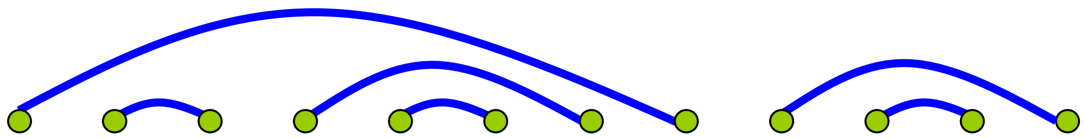
Any two particles connected by a bond form a singlet

Complete, linearly independent basis for the space of all singlet states.

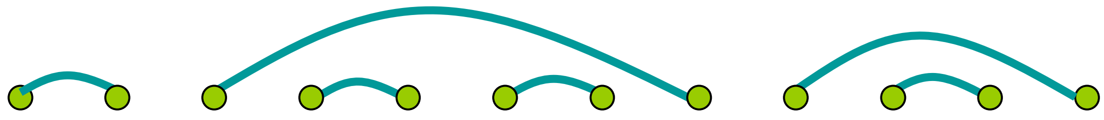
Valence Bonds Basis

Nonorthogonal basis, but easy to compute with:

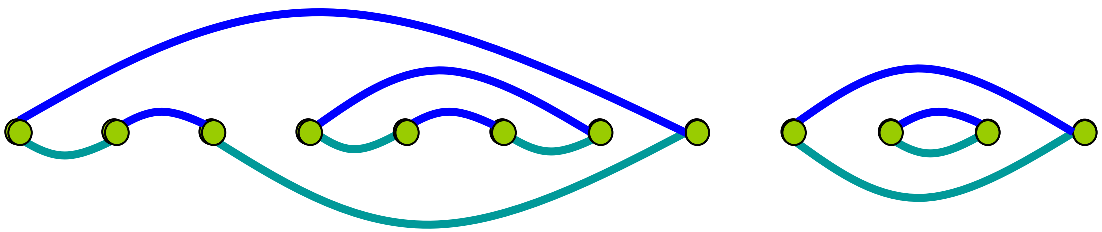
$|\alpha\rangle =$



$|\beta\rangle =$

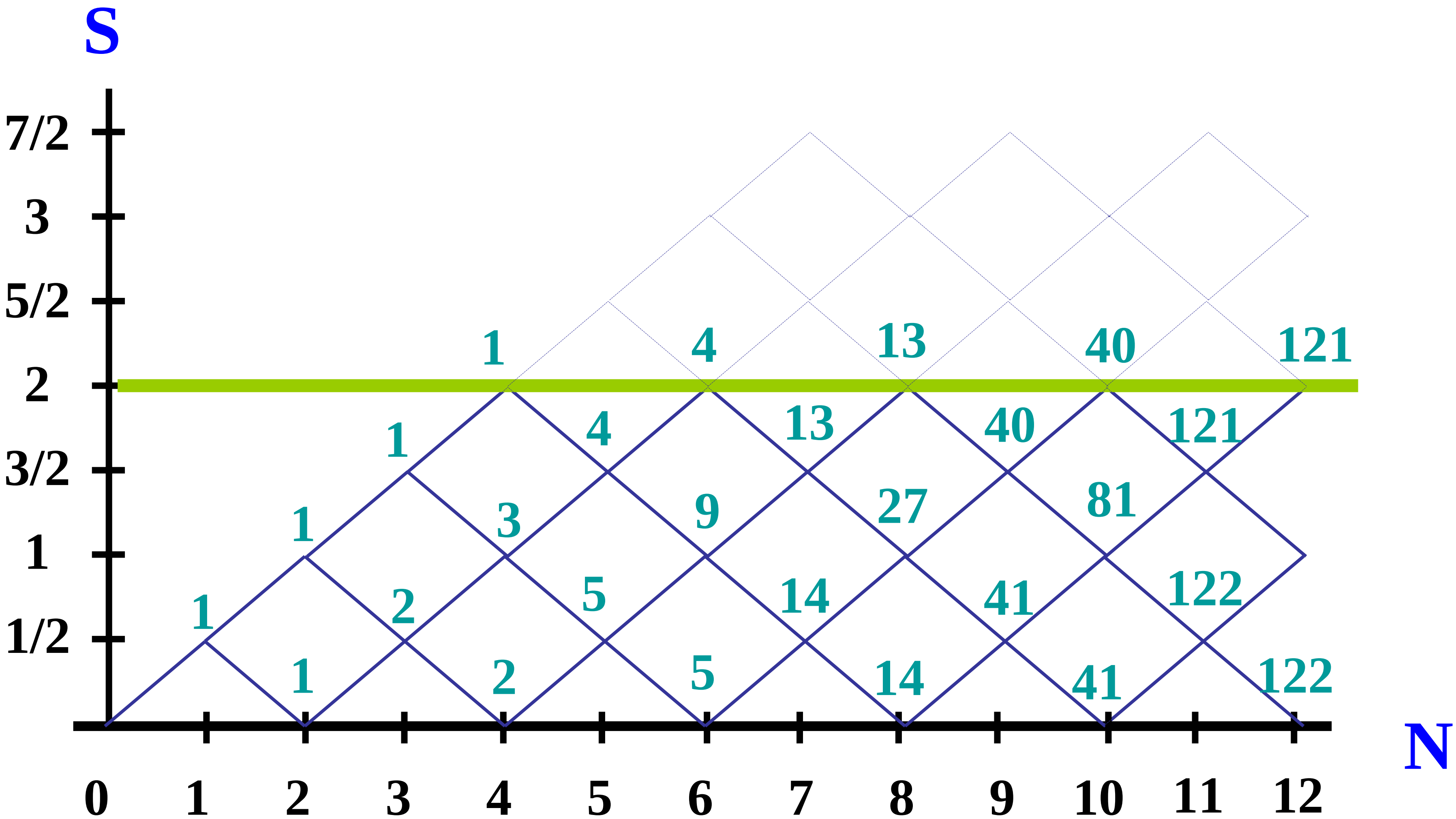


$\langle\alpha|\beta\rangle =$



$$\langle\alpha|\beta\rangle = 2^{N_{loops} - N/2} = 2^{3-12/2} = 1/8$$

$$k = 4$$


S
7/2
3
5/2
2
3/2
1
1/2
N
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3
4
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6
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8
9
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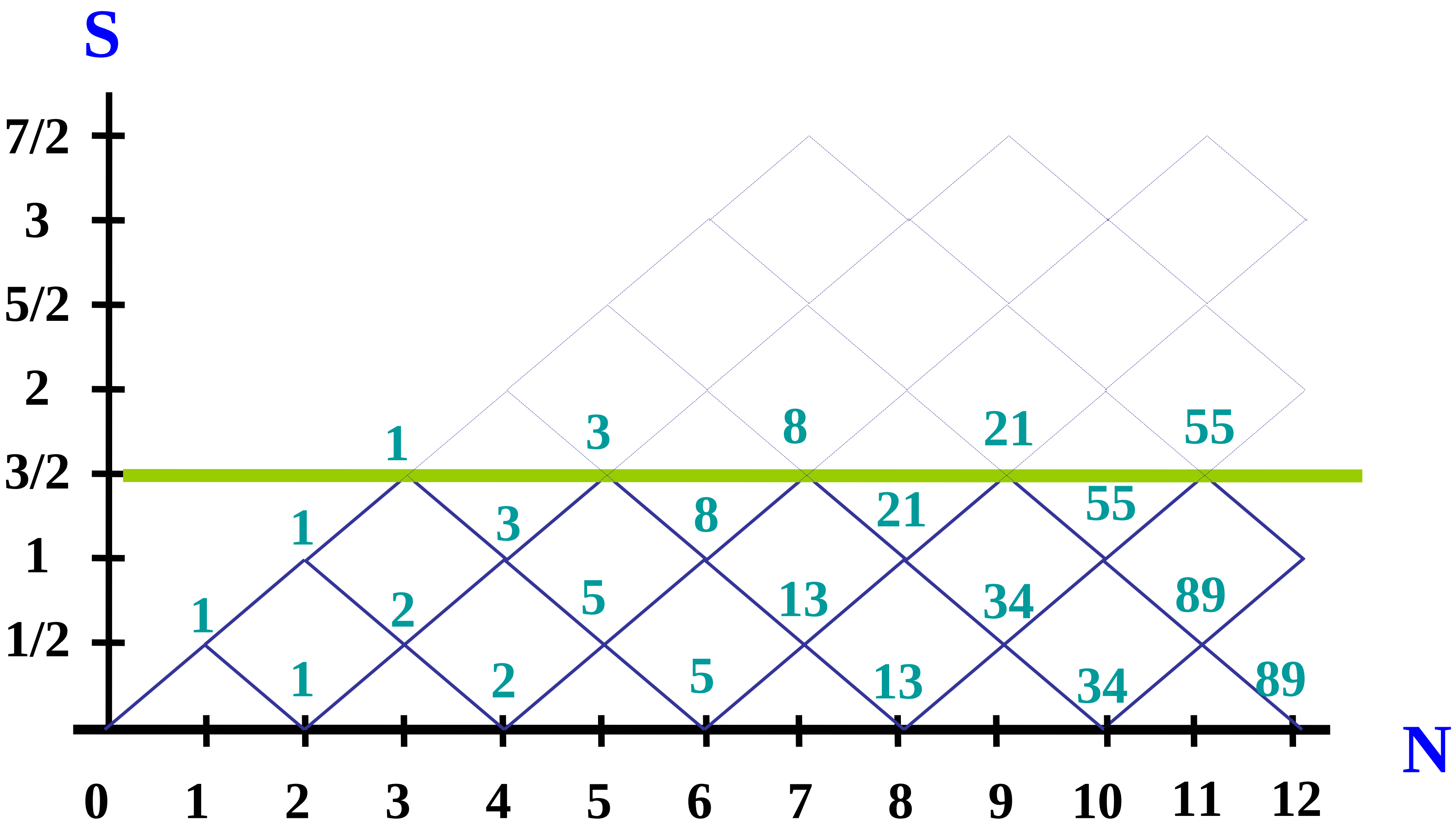
11
12
1
4
13
40
121
1
4
13
40
121
1
3
9
27
81
1
2
5
14
41
122
1
2
5
14
41
122

Dim(N) ~ $3^{N/2}$

$k = 3$

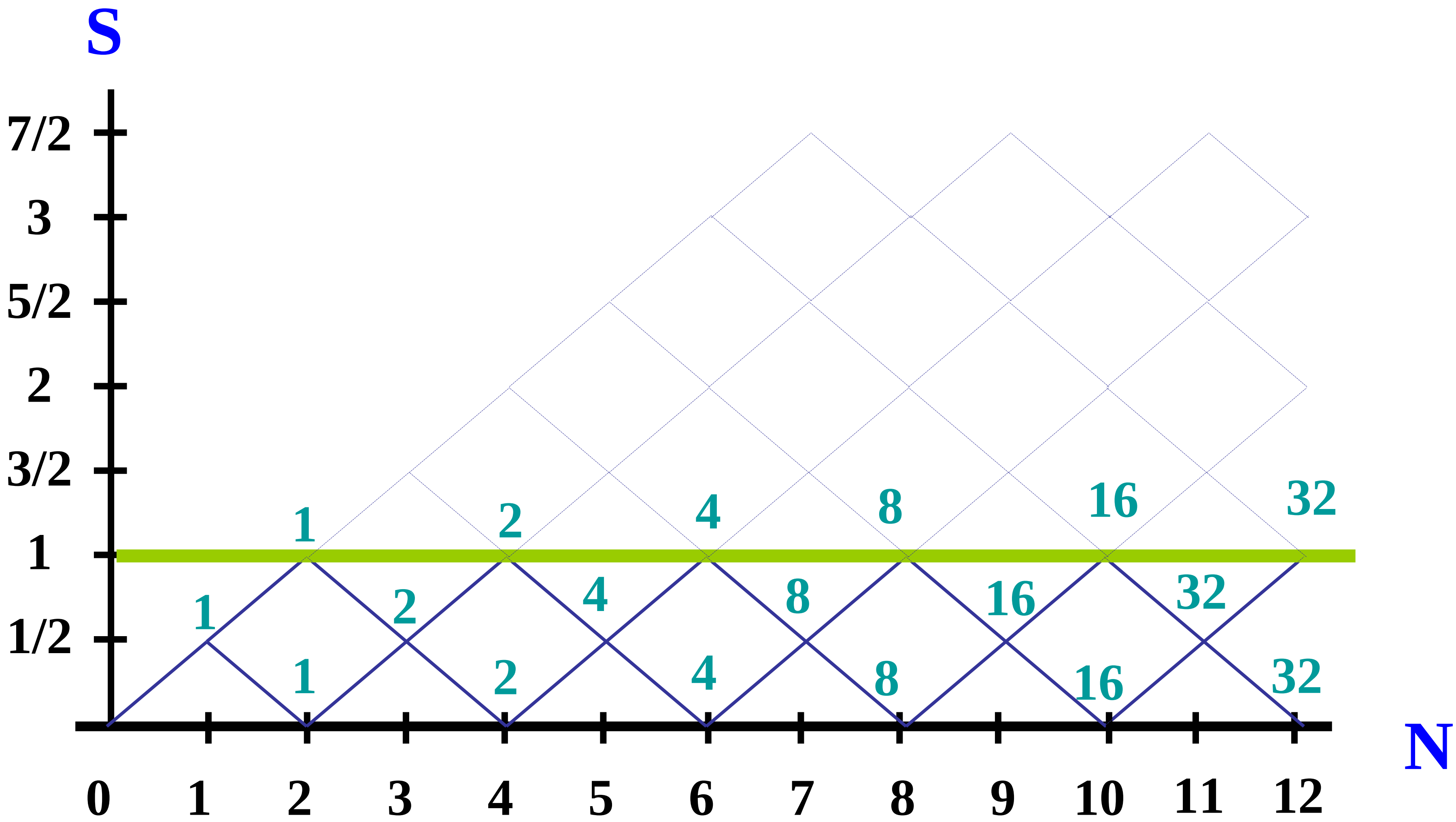
(ν=12/5 state?)



Dim(N) = Fib(N+1) ~ ϕ^N

$k = 2$

(ν=5/2 state)



Dim(N) = $2^{N/2-1}$

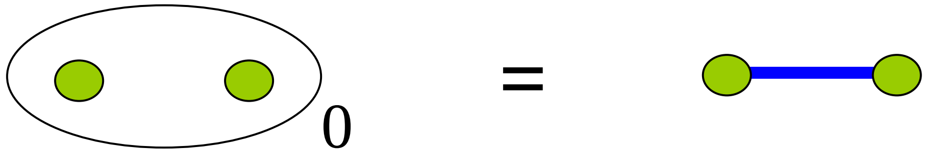
Quantum Dimension

Hilbert space of N particles with topological charge ½ grows asymptotically as $\boldsymbol{d}^{N}$ where $\boldsymbol{d}$ is the “quantum dimension” of the particles.

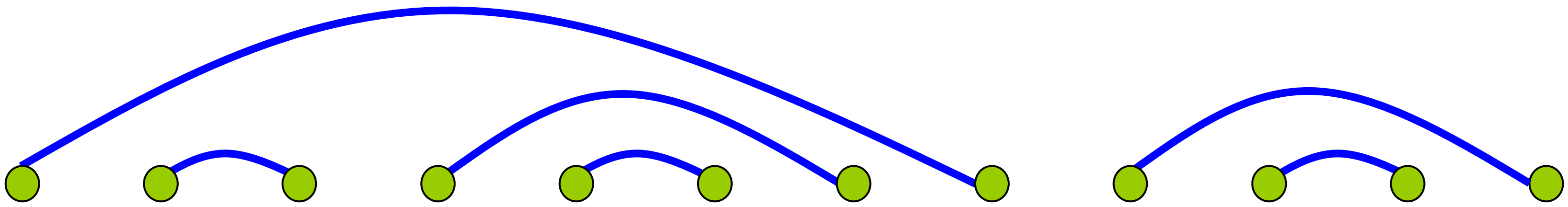
$$d = 2\cos\frac{\pi}{k+2}$$

k	d
2	$\sqrt{2}$
3	$\phi = \frac{1+\sqrt{5}}{2}$
4	$\sqrt{3}$
$\vdots$	$\vdots$
∞	2

Valence Bonds Basis for $SU(2)_k$



Non-crossing valence bond basis:



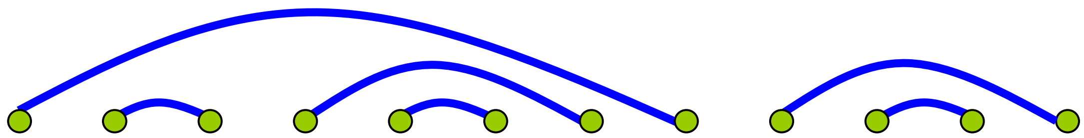
Any two particles connected by a bond fuse to trivial topological charge 0 if brought together.

A complete, but linearly dependent basis for the space of all states with total topological charge 0.

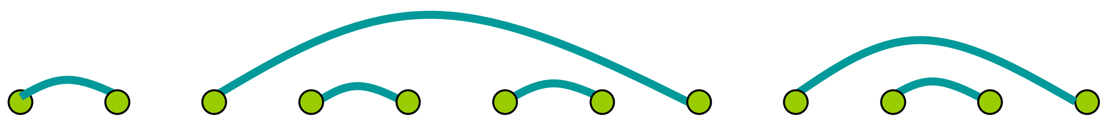
Valence Bonds Basis for $SU(2)_k$

Again, nonorthogonal, but still easy to compute with:

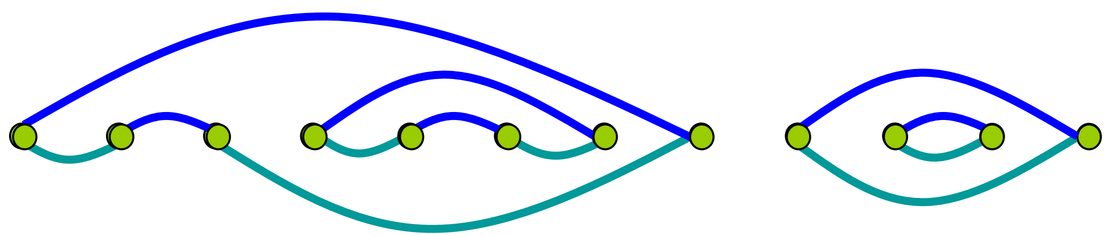
$|\alpha\rangle =$



$|\beta\rangle =$



$\langle\alpha|\beta\rangle =$

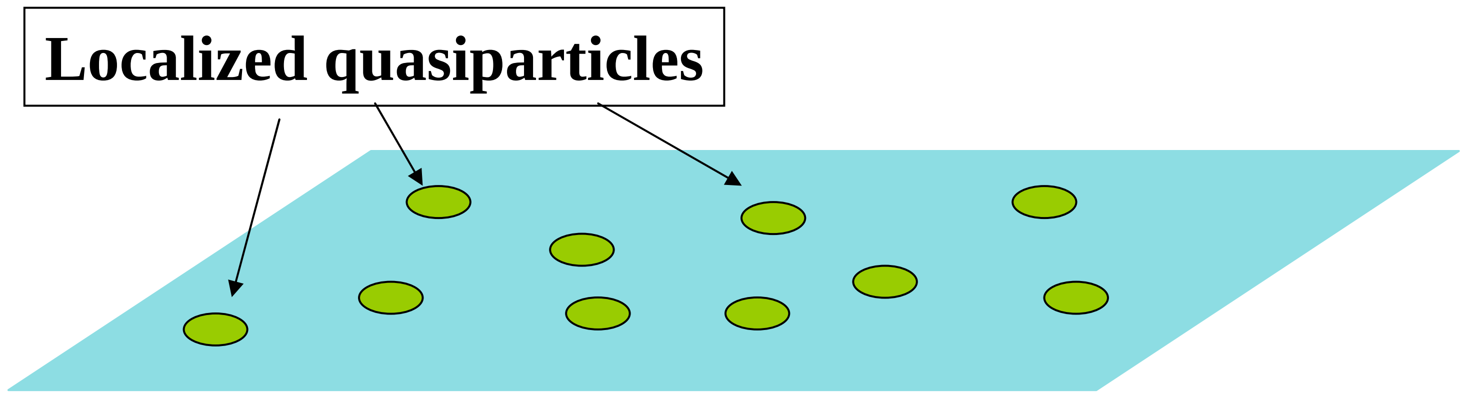


$$\langle\alpha|\beta\rangle = \boldsymbol{d}^{N_{loops} - N/2} = \boldsymbol{d}^{4-12/2} = 1/\boldsymbol{d}^2$$

$$d = 2\cos\frac{\pi}{k+2}$$

Quantum Dimension

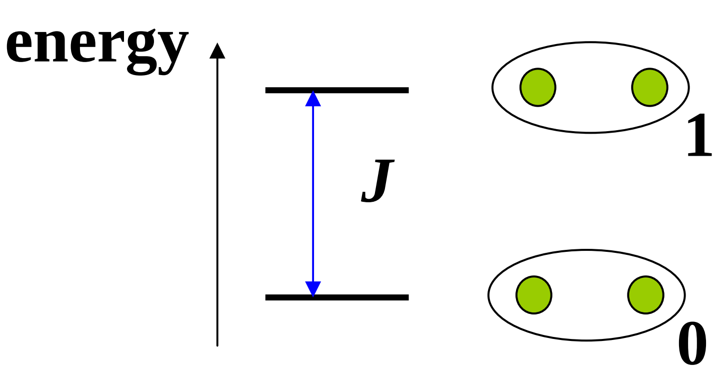
Interacting Non-Abelian Anyons



Topological degeneracy is lifted when quasiparticles are close together (for FQHE states, this means within a few magnetic lengths).

Assume trivial topological charge is energetically favored:

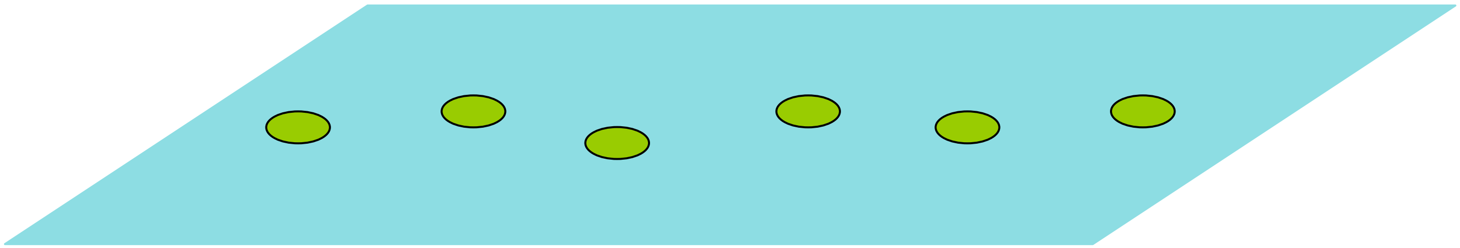
$$H = -\sum_{i,j} J_{i,j}\ \Pi^{0}_{i,j} \quad ; \quad J_{i,j} > 0$$



Projection onto state of particle *i* and *j* with total topological charge *0*.

Interacting Anyon Chain

(Feiguin et al., PRL 98, 160409 (2007).)



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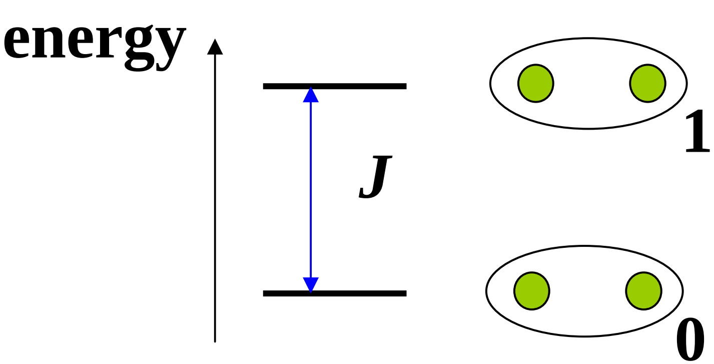

$$H = -\sum_i J_i \Pi_i^0 \; ; \quad J_i > 0$$

$$(\; \Pi_i^0 \equiv \Pi_{i,i+1}^0 \;)$$

Uniform SU(2)$_k$ Chains

$k \to \infty$ Ordinary spin-1/2 AFM Heisenberg model:

$$H = -J \sum_i \Pi_i^0 = -J \sum_i \left(\frac{1}{4} - S_i \cdot S_{i+1} \right)$$

Conformally invariant quantum critical model with central charge: c=1

Uniform SU(2)$_k$ chains can be mapped onto exactly solvable Andrews-Baxter-Forrester models which realize minimal CFTs with central charges,

$$c = 1 - \frac{6}{(k+1)(k+2)}$$

(Feiguin et al., PRL 98, 160409 (2007).)

$k \to \infty$	$c = 1$	Heisenberg Model
$k = 3$	$c = \frac{7}{10}$	Golden Chain
$k = 2$	$c = \frac{1}{2}$	Critical TFIM

Random $SU(2)_k$ Chains

$$H = -\sum_i J_i \Pi_i^0$$

Given the similarity between ordinary spin and $SU(2)_k$ particles we can apply the real space RG. (Ma, DasGupta, Hu '79, D. Fisher '94)

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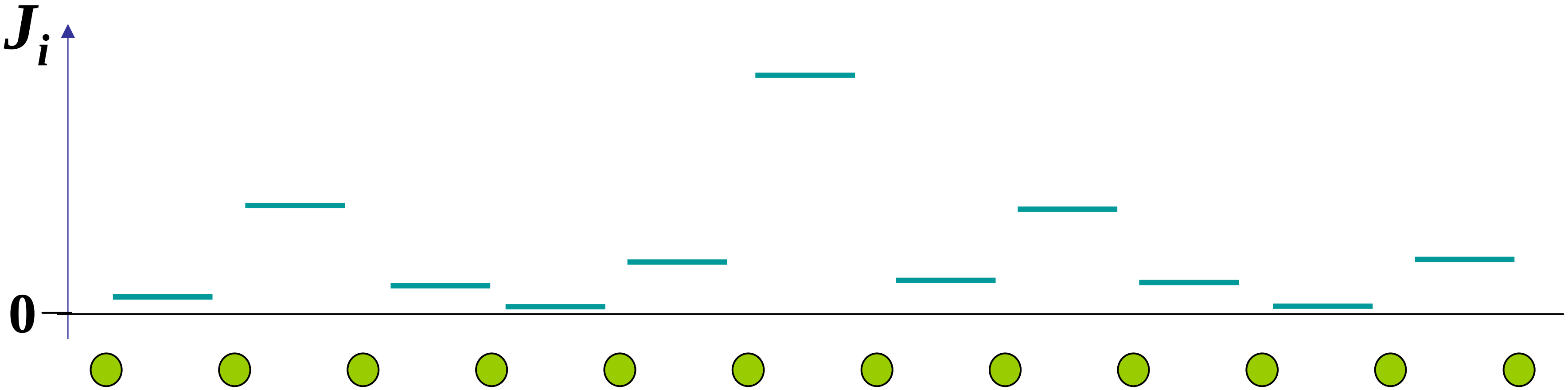

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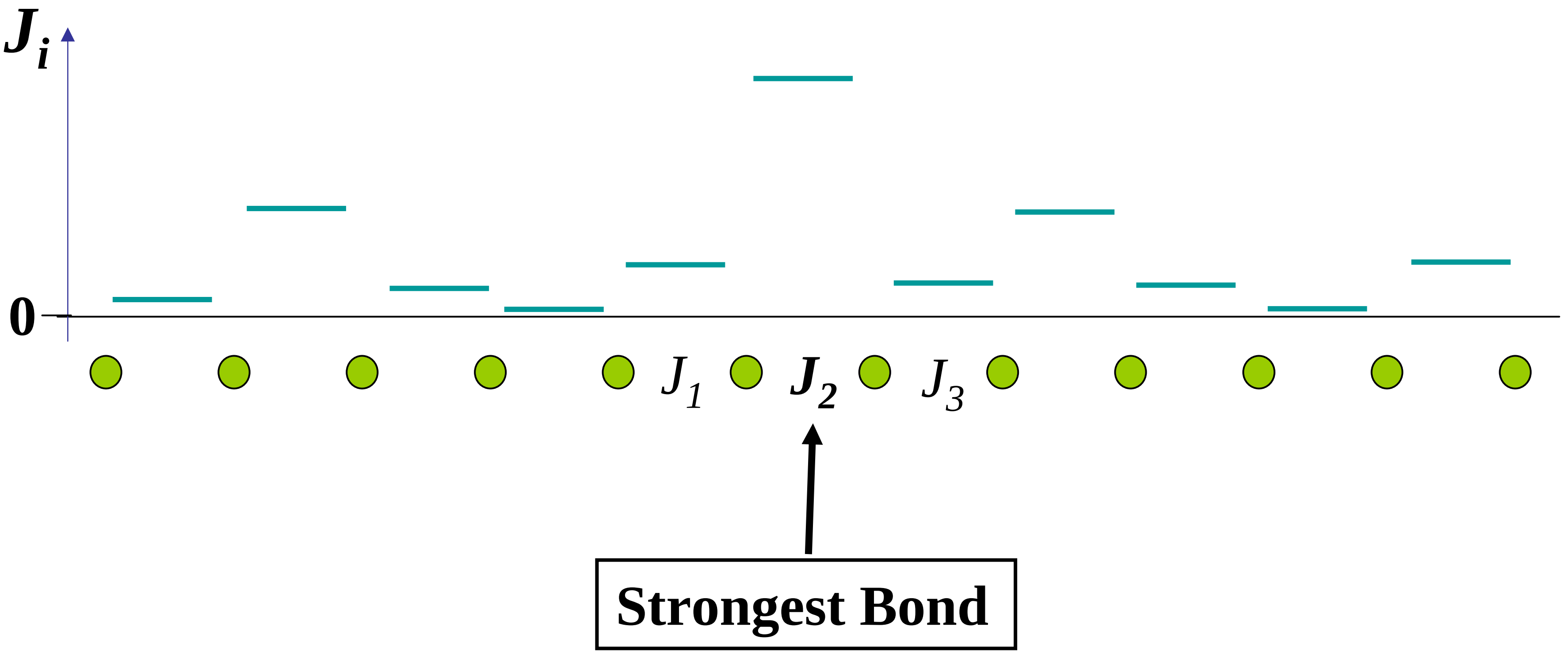

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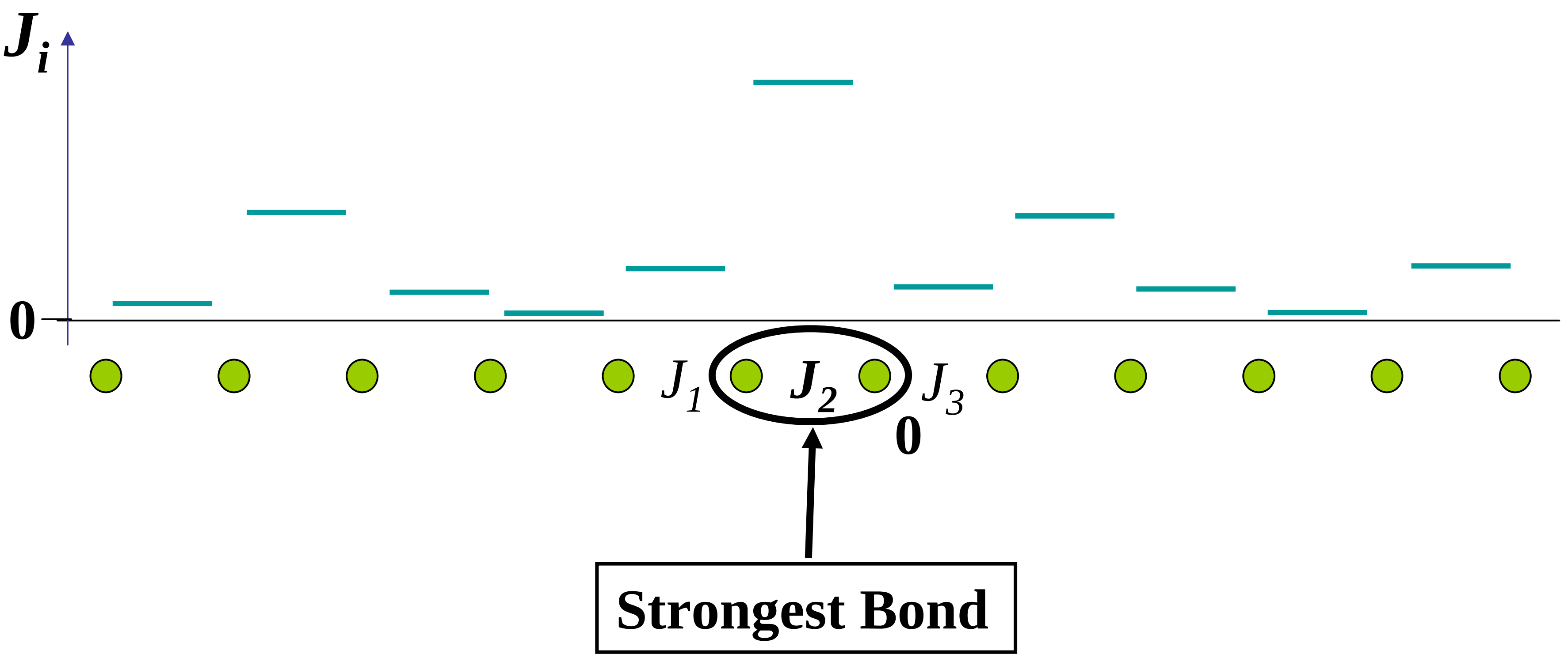

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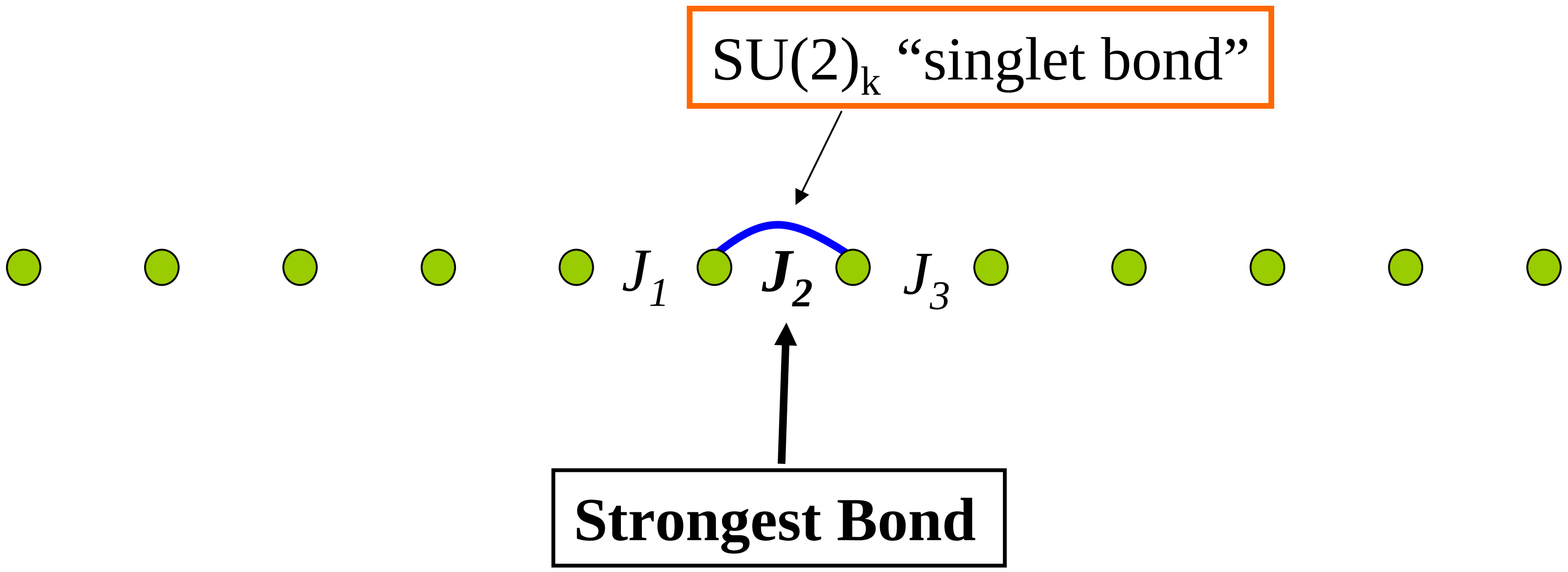

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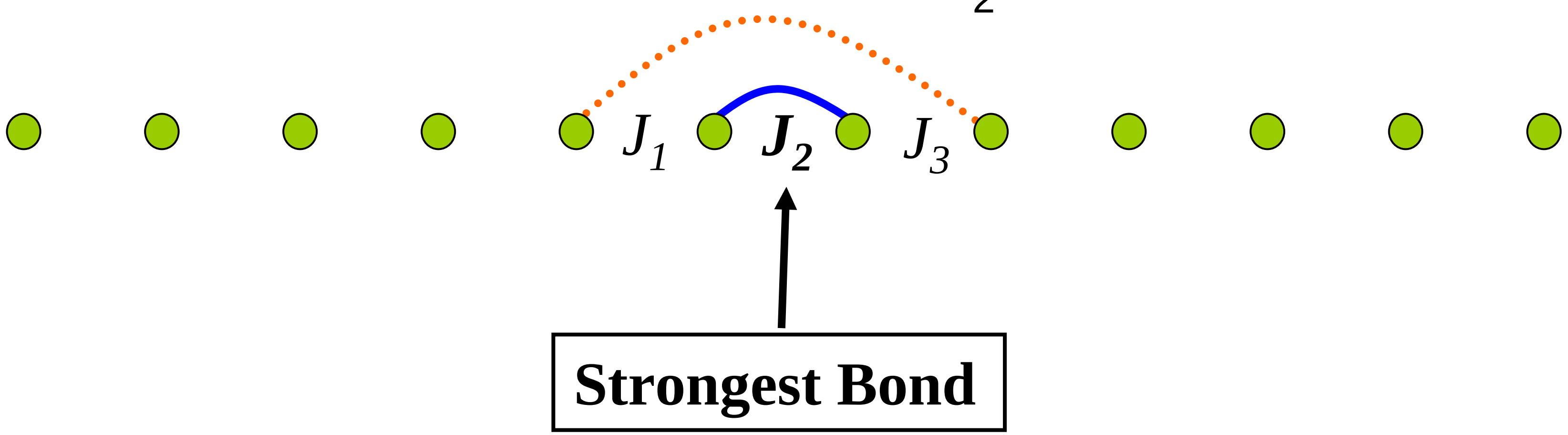
Effective interaction from 2nd order perturbation theory

$$\tilde{J} = \frac{1}{2} \frac{J_1 J_3}{J_2}$$

Spin-1/2 particles
(Ma, DasGupta, Hu '79)



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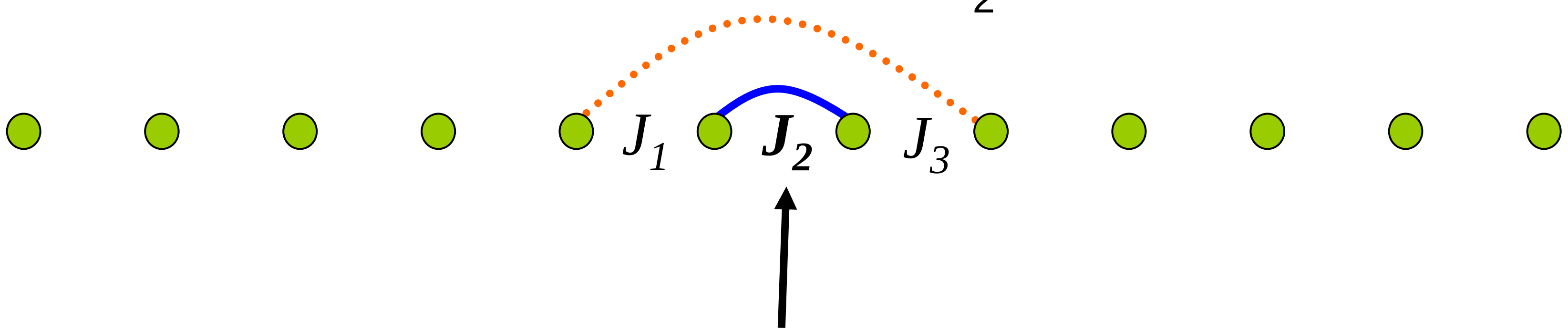
Effective interaction from 2nd order perturbation theory

$$\tilde{J} = \frac{2}{d^2}\frac{J_1 J_3}{J_2}$$

$SU(2)_k$ particles (NEB, K.Yang, PRL'07)



Strongest Bond

Random SU(2)$_k$ Chains

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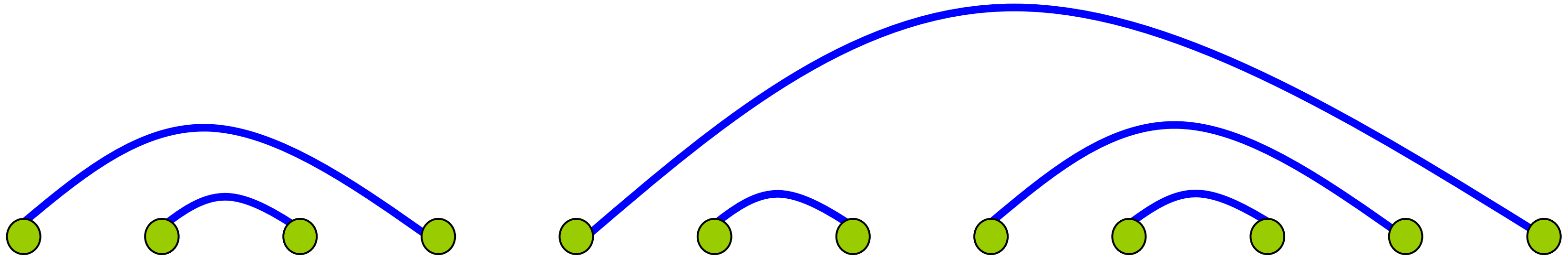
Given the similarity between ordinary spin and $SU(2)_k$ particles we can apply the real space RG. (Ma, DasGupta, Hu '79, D. Fisher '94)



Random Singlet Phase for $SU(2)_k$ particles: Bonds freeze into a particular non-crossing valence-bond state.

(NEB, K.Yang, PRL '07)

Random $SU(2)_k$ Chains

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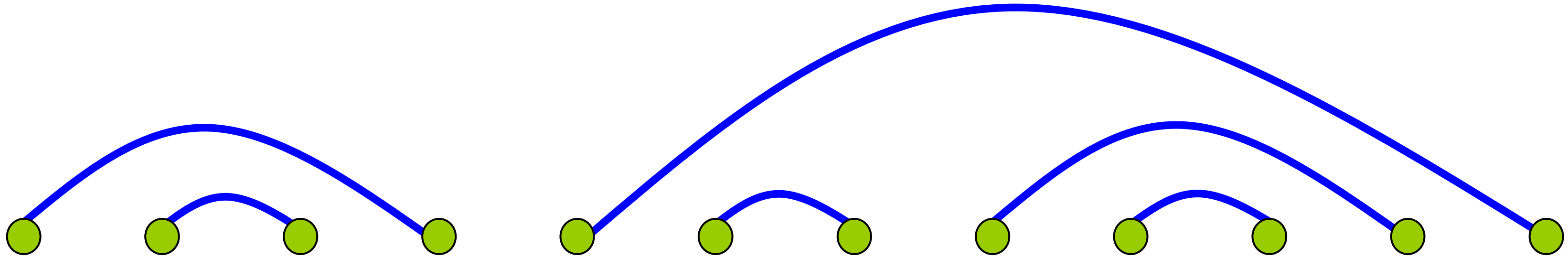

Infinite Random Fixed Point (D. Fisher '94)

$$L^{1/2} \sim \ln \frac{1}{E} \quad \longrightarrow \quad \textbf{Specific Heat:} \quad C \propto \frac{1}{|\ln T|^3}$$

Random $SU(2)_k$ Chains

$$H = -\sum_i J_i \Pi_i^0$$

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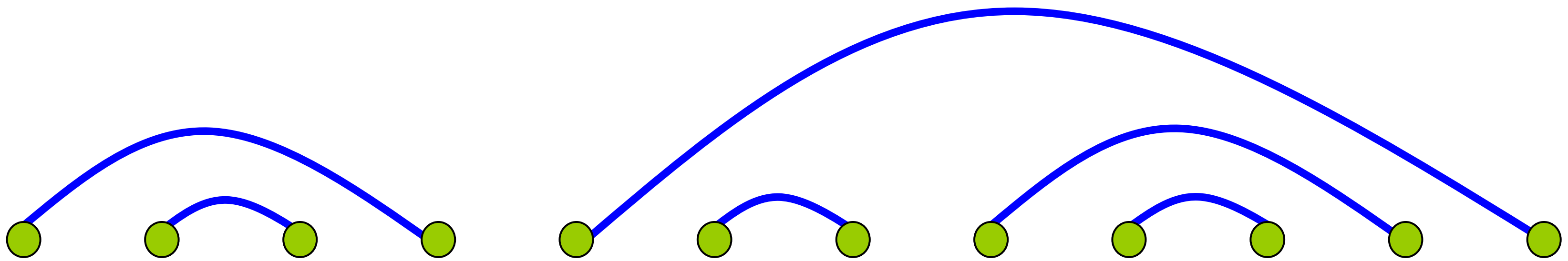


w/ FM bonds (i.e. some $J_i < 0$), for k=3 → New fixed point

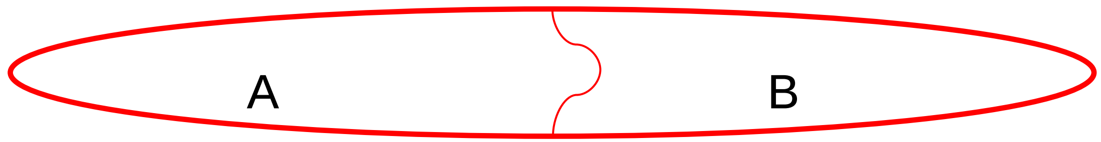
$$L^{1/3} \sim \ln \frac{1}{E} \rightarrow$$

Specific Heat: $C \propto \frac{1}{|\ln T|^4}$

(L. Fidkowski, G. Refael, NEB, J. Moore, arxiv:0807.1123)

Entanglement Entropy

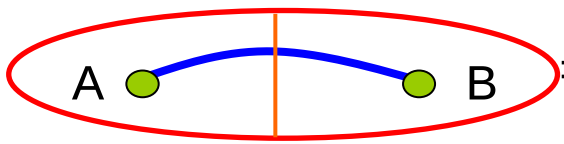
A quantum system composed of two parts: A and B



$$\rho_A = \mathrm{Tr}_B\left[|GS\rangle\langle GS|\right] \quad \longrightarrow \quad S_A \equiv -\mathrm{Tr}\left[\rho_A \log_2 \rho_A\right]$$

Reduced density matrix **Entanglement entropy**

Simple example: An *SU(2)* singlet bond

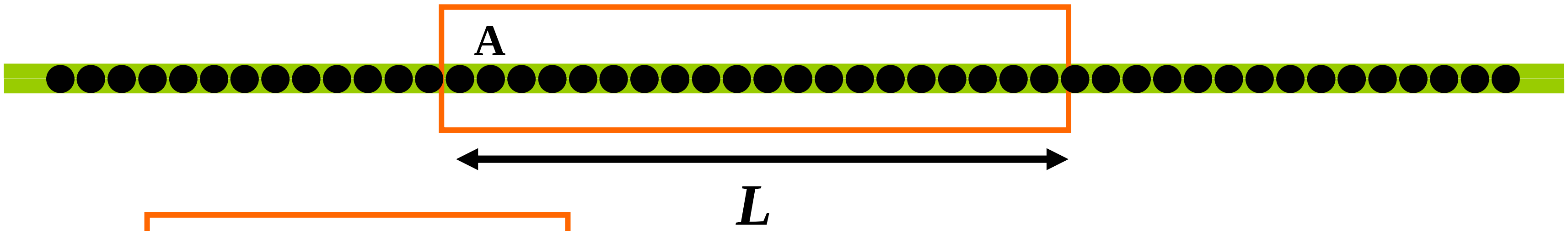


$$= \frac{1}{\sqrt{2}}\left(\uparrow_A \downarrow_B - \downarrow_A \uparrow_B\right)$$

$$\rho_A = \frac{1}{2}\left|\uparrow_A\right\rangle\left\langle\uparrow_A\right| + \frac{1}{2}\left|\downarrow_A\right\rangle\left\langle\downarrow_A\right| \quad \longrightarrow \quad \boxed{S_A = 1}$$

Entanglement Entropy

At 1+1 dimensional conformally invariant quantum critical points, the entanglement entropy scales logarithmically with the size of region A with a universal coefficient:



$$S(L) \approx \frac{c}{3} \log_2 L$$

c = central charge

(Holzhey et al. ‘94, Calabrese & Cardy ‘04)

$$S(L) \approx \frac{1}{3} \log_2 L$$

For uniform Heisenberg model (c=1)

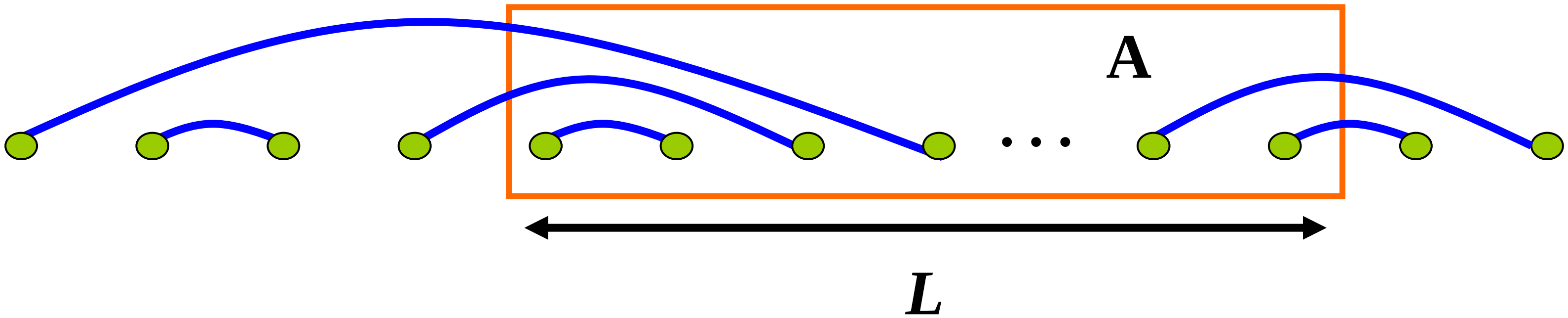
$$S(L) \approx \frac{1}{6} \log_2 L$$

For uniform critical TFIM (c=1/2)

Entanglement Entropy of Random Spin-1/2 Chains

(Refael & Moore PRL 93, 260602 (2004))

In the random singlet phase the entanglement entropy also scales logarithmically with ***L***



Avg. # of bonds leaving region of length L $\approx \frac{1}{3}\ln L = \frac{\ln 2}{3}\log_2 L$

$$S(L) \approx \underbrace{(entropy\ per\ bond)}_{1} \times \frac{\ln 2}{3}\log_2 L \approx \frac{\ln 2}{3}\log_2 L$$

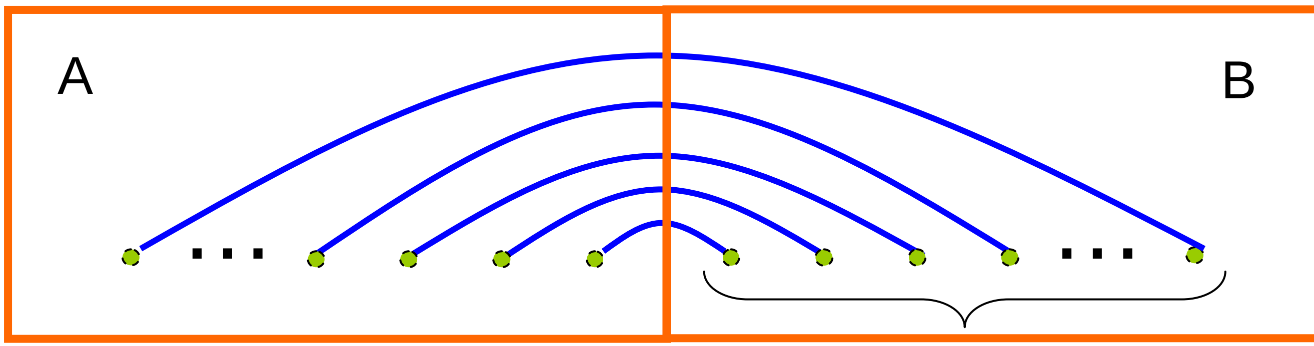
→ **"effective" central charge:** $\tilde{c} = \ln 2$

Entanglement Entropy of $SU(2)_k$ Singlet Bond

(NEB & Kun Yang, PRL 99, 140405 (2007))

For $SU(2)_k$ random chains the only thing that is different is the entanglement per bond.

Imagine N >> 1 “singlet” pairs:



N particles
Dimensionality of Hilbert space ~ $d^{\,N}$

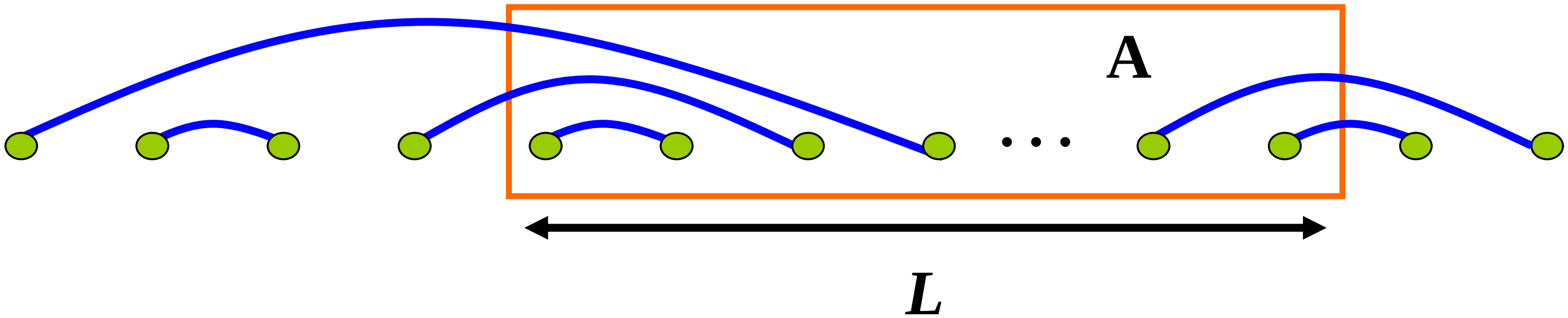
$$S_A \approx \log_2 d^{\,N} = N \log_2 d$$

Entropy per bond = $\log_2 d$

Entanglement Entropy of Random $SU(2)_k$ Chains

(NEB & Kun Yang, PRL 99, 140405 (2007))

In the random singlet phase the entanglement entropy also scales logarithmically with ***L***



Avg. # of bonds leaving region of length L $\approx \frac{1}{3}\ln L = \frac{\ln 2}{3}\log_2 L$

$$S(L) \approx \underbrace{(entropy\ per\ bond)}_{\log_2 d} \times \frac{\ln 2}{3}\log_2 L \approx \frac{\ln d}{3}\log_2 L$$

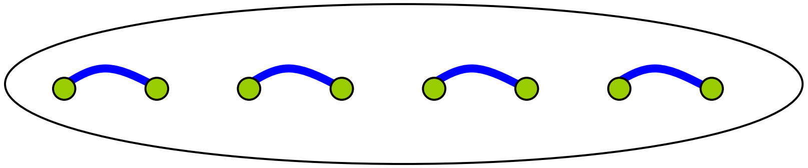
→ **"effective" central charge:** $\tilde{c} = \ln d$

Valence-Bond Monte Carlo

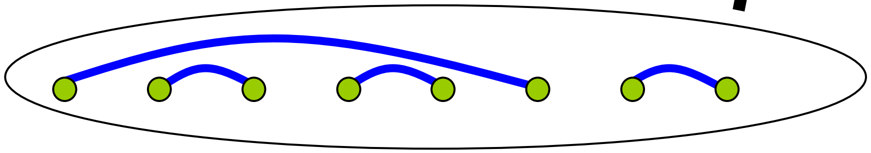
(Sandvik, PRL **95**, 207203 (2005))

Idea: Project out ground state of *H* by repeatedly applying –*H* to some initial valence-bond state $|S_0>$

$$(-H)^n |S_0\rangle = \sum_{i_1 \cdots i_n} J_{i_1} \cdots J_{i_n} \, \Pi^{\mathbf{0}}_{i_1} \cdots \Pi^{\mathbf{0}}_{i_n} |S_0\rangle = \sum_{\alpha} w(\alpha) |\alpha\rangle$$



Initial valence-bond state



Sum over “non-crossing” valence-bond states.

Weight factors $w(\alpha)$ are easy to compute and update for efficient Monte Carlo sampling. Straightforward to generalize to $SU(2)_k$ particles.

Valence-Bond Entanglement

(Alet, Capponi, Laflorencie, Matthieu, PRL 99, 117204 (2007))

For the ground state wavefunction $|GS\rangle = \sum w(\alpha)|\alpha\rangle$

the "valence-bond entanglement" is defined to be:

$$S_{VB}(L) = \frac{\sum_{\alpha} w(\alpha)\, S(L;\alpha)}{\sum_{\alpha} w(\alpha)}$$

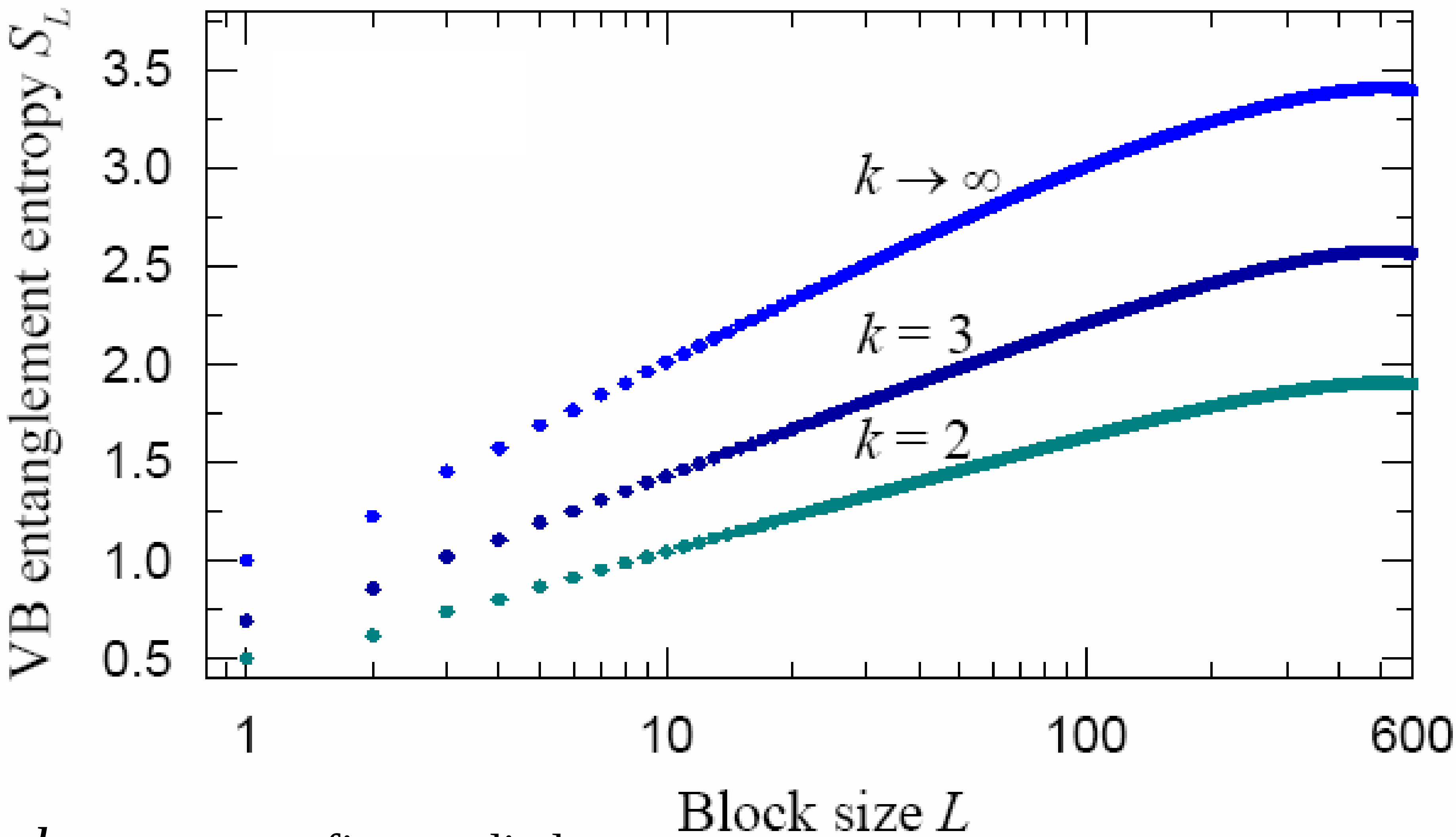
Entanglement entropy in the valence-bond state $|\alpha>$ computed a la Refael and Moore.

Exact result for uniform chains (Jacobsen & Saleur, PRL 100, 087205 (2008))

$$S_{VB}(L) \approx \frac{4\ln d}{\pi^2}\frac{1}{k+1}\frac{d}{\sqrt{4-d^2}}\log_2 L$$

Close to, but not exactly equal to c/3

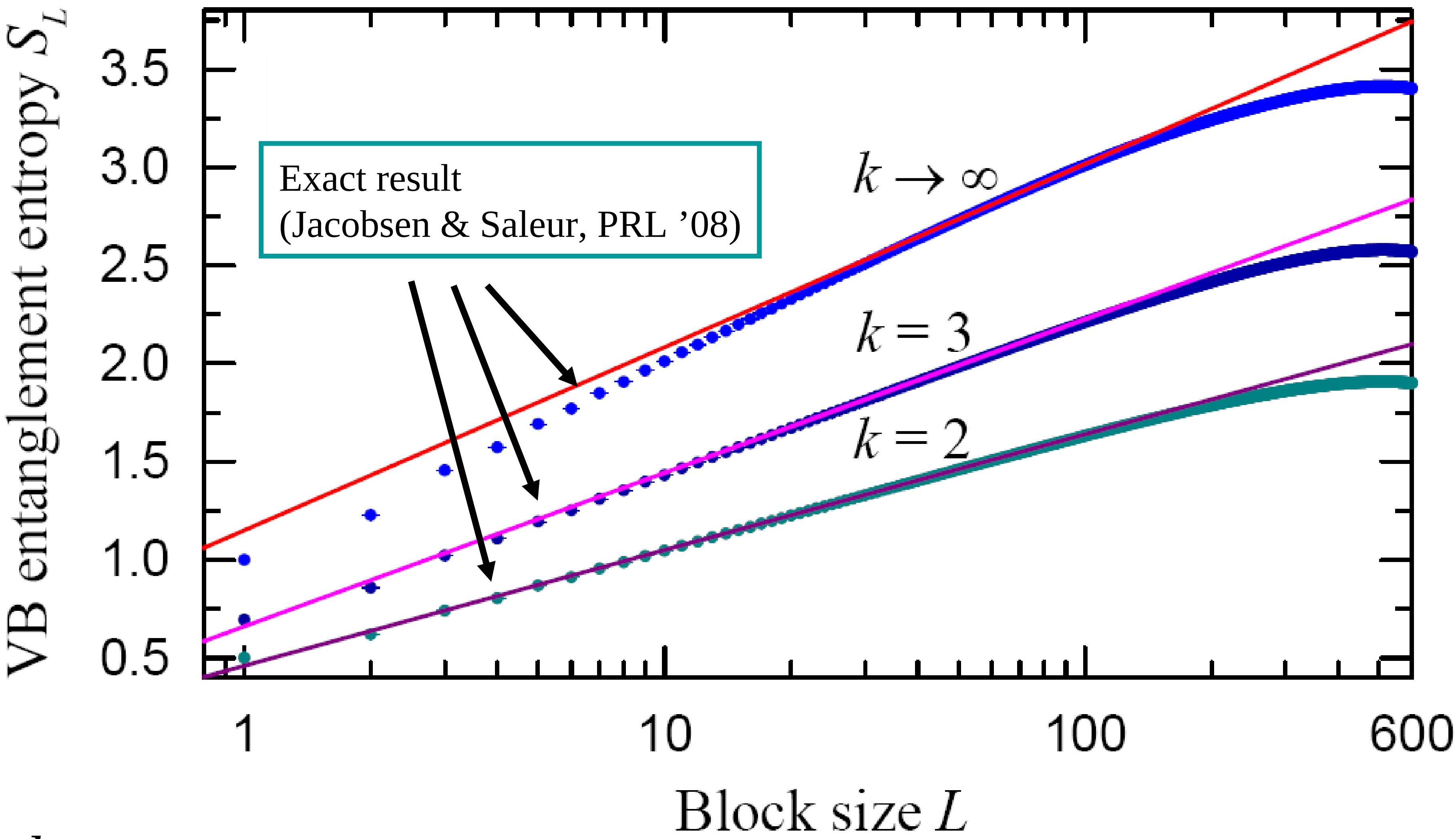
Valence-Bond Entanglement: Uniform Case



$k \to \infty$ case first studied by Alet et al. '07

H. Tran, NEB, in preparation

Valence-Bond Entanglement: Uniform Case



$k \to \infty$ case first studied by Alet et al. '07

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Valence-Bond Entanglement

(Alet, Capponi, Laflorencie, Matthieu, PRL 99, 117204 (2007))

For the ground state wavefunction $|GS\rangle = \sum w(\alpha)|\alpha\rangle$

the "valence-bond entanglement" is defined to be:

$$S_{VB}(L) = \frac{\sum_{\alpha} w(\alpha)\, S(L;\alpha)}{\sum_{\alpha} w(\alpha)}$$

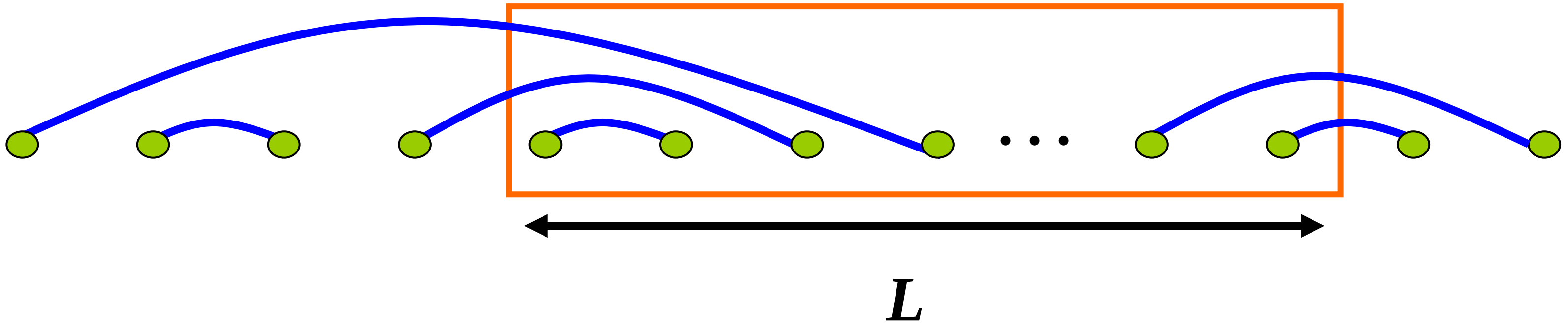
Entanglement entropy in the valence-bond state $|\alpha>$ computed a la Refael and Moore.

If bonds "freeze" on long length scales then $S_{VB}(L)$ should show the same scaling as the "true" entanglement S(L) for large L.

→ **For random chains expect:** $S_{VB}(L) \approx \frac{\ln d}{3} \log_2 L$

For random chains, how do we know bonds are "freezing"?

Look at fluctuations in number of bonds leaving region of size L.



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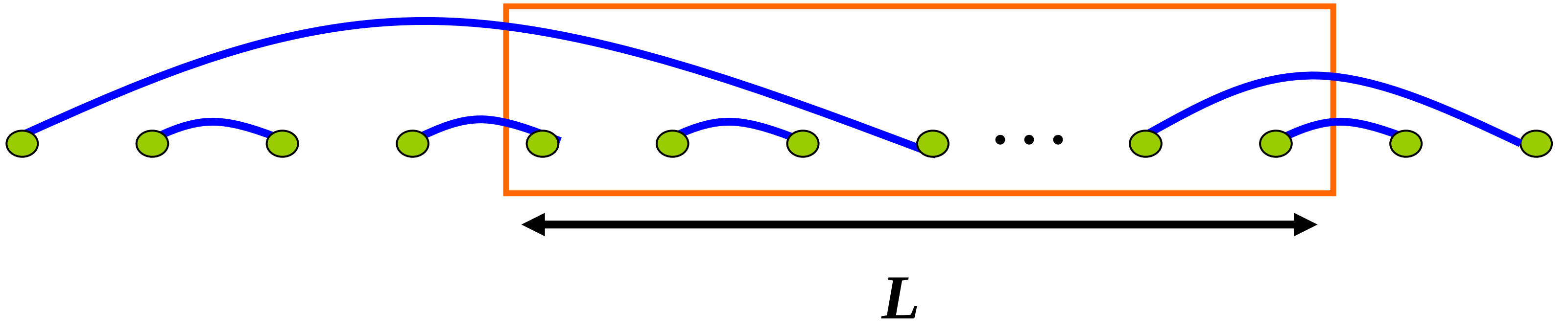

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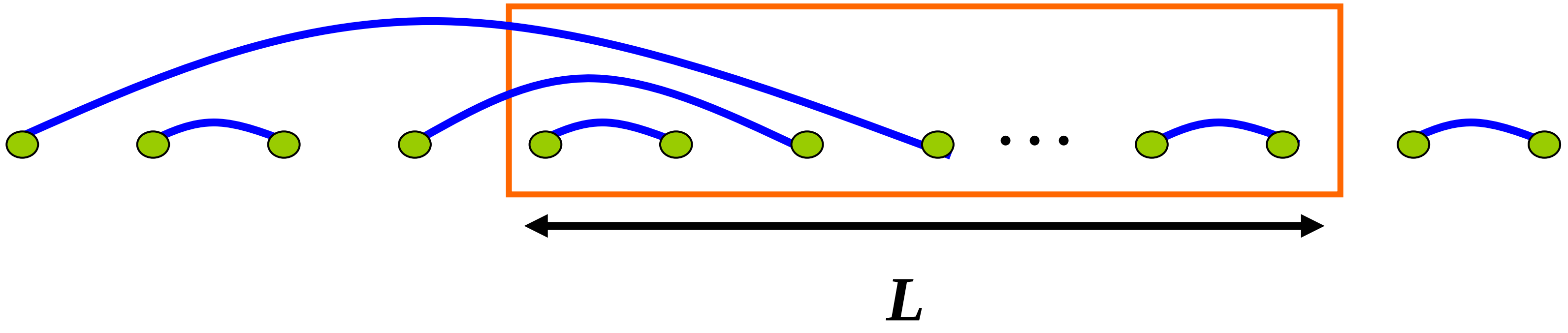

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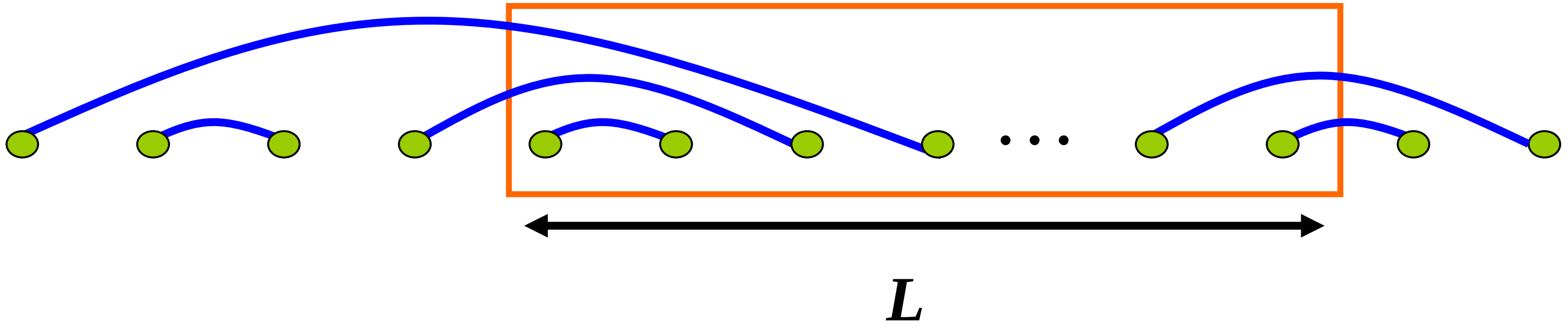

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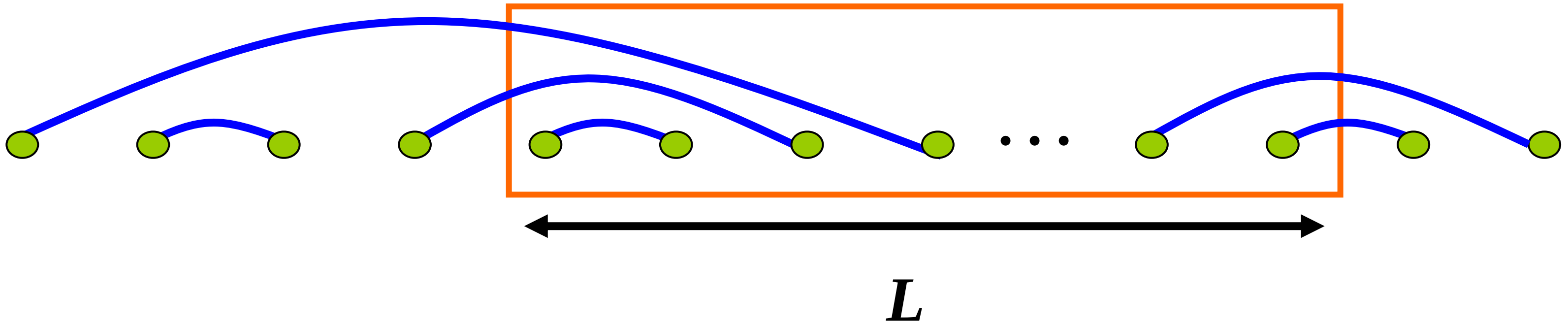


If bonds are frozen, only fluctuations near boundary of region change the number of bonds leaving that region.

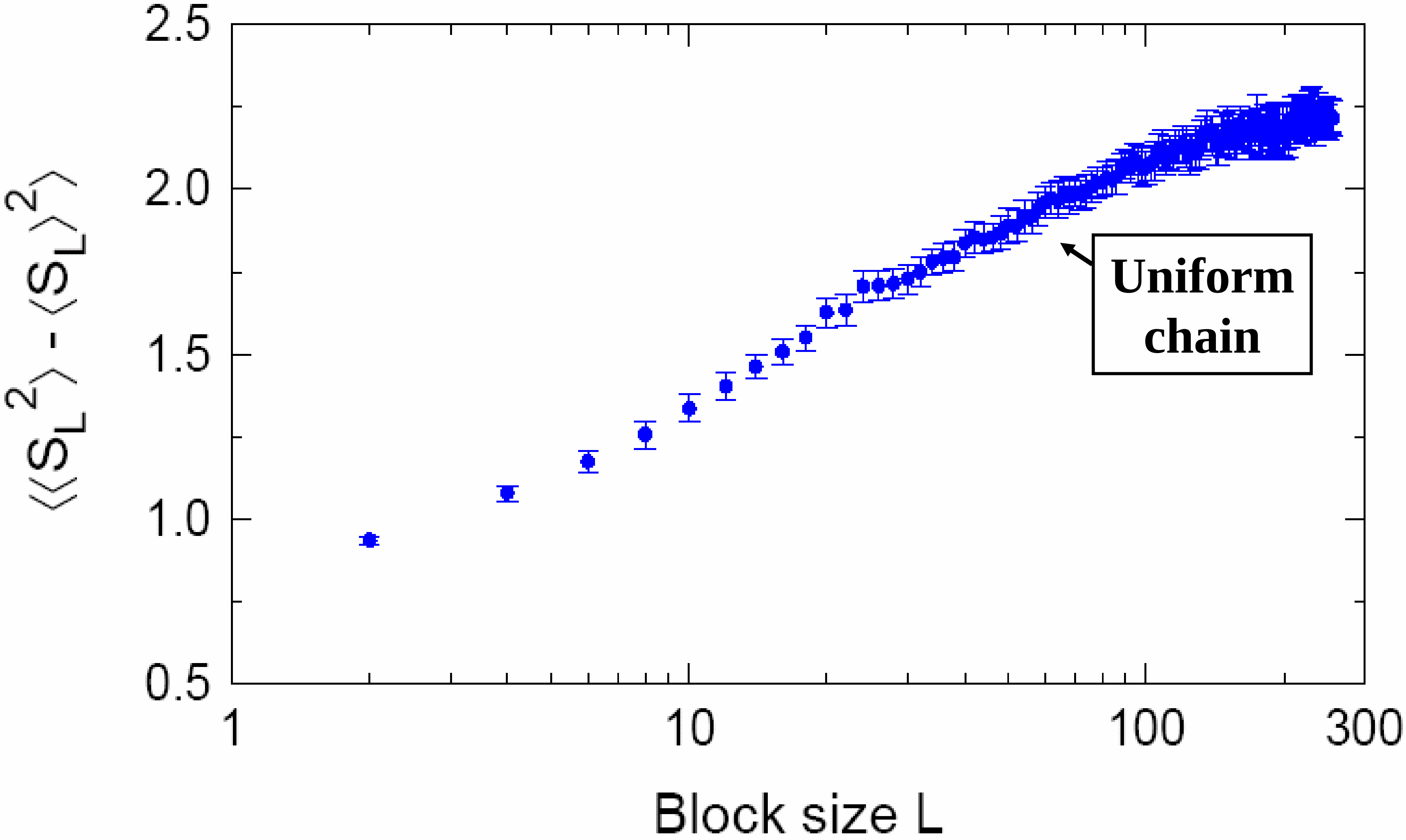
$$\sigma_n^2 = \left\langle \left\langle n_L{}^2 \right\rangle - \left\langle n_L \right\rangle^2 \right\rangle$$

Average over disorder

Bond fluctuations for particular realization of disorder

Expect $\sigma_n{}^2$ to be independent of L for large L if bonds freeze.

Bond Fluctuations: Signature of Freezing



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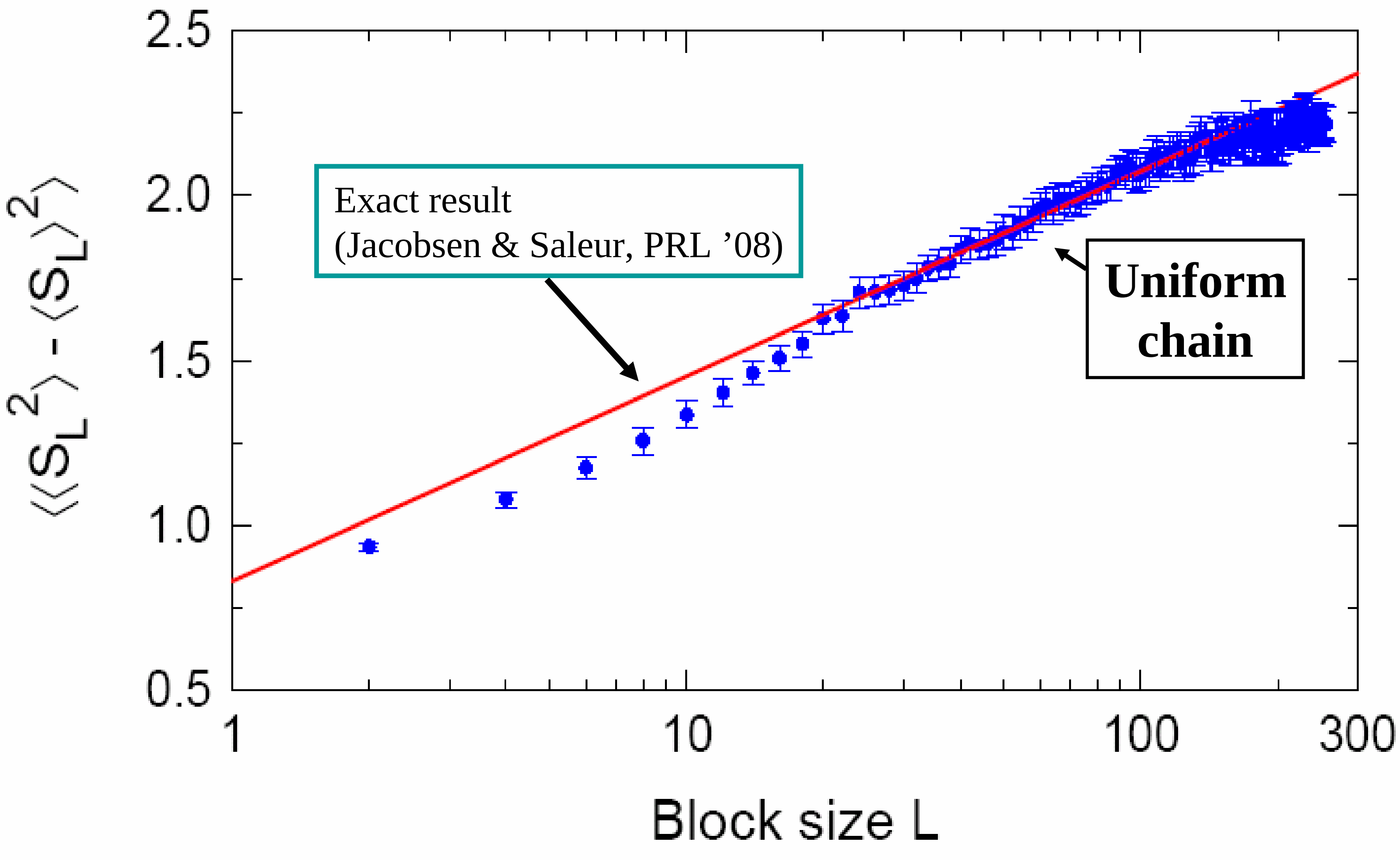
Bond Fluctuations: Signature of Freezing



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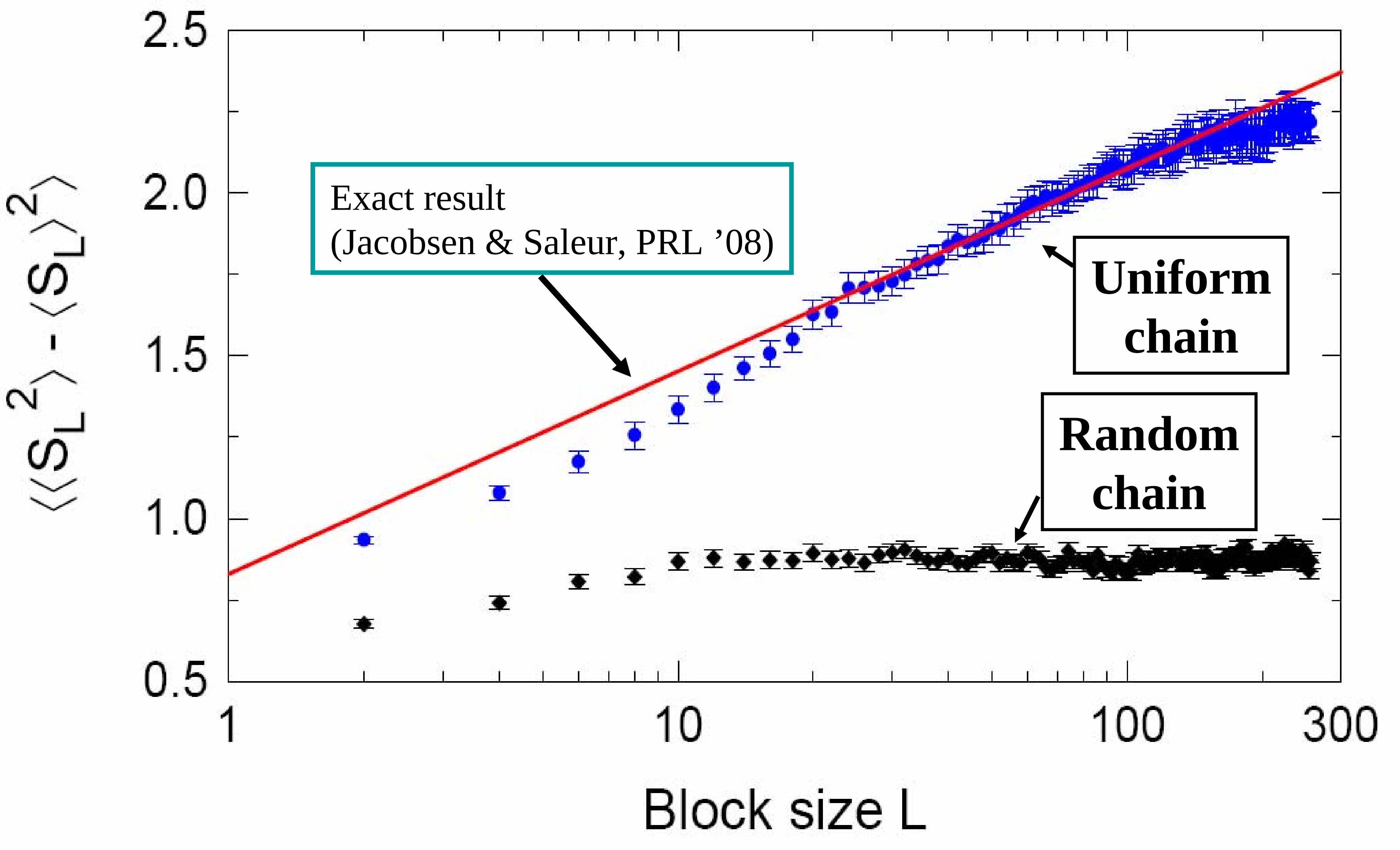

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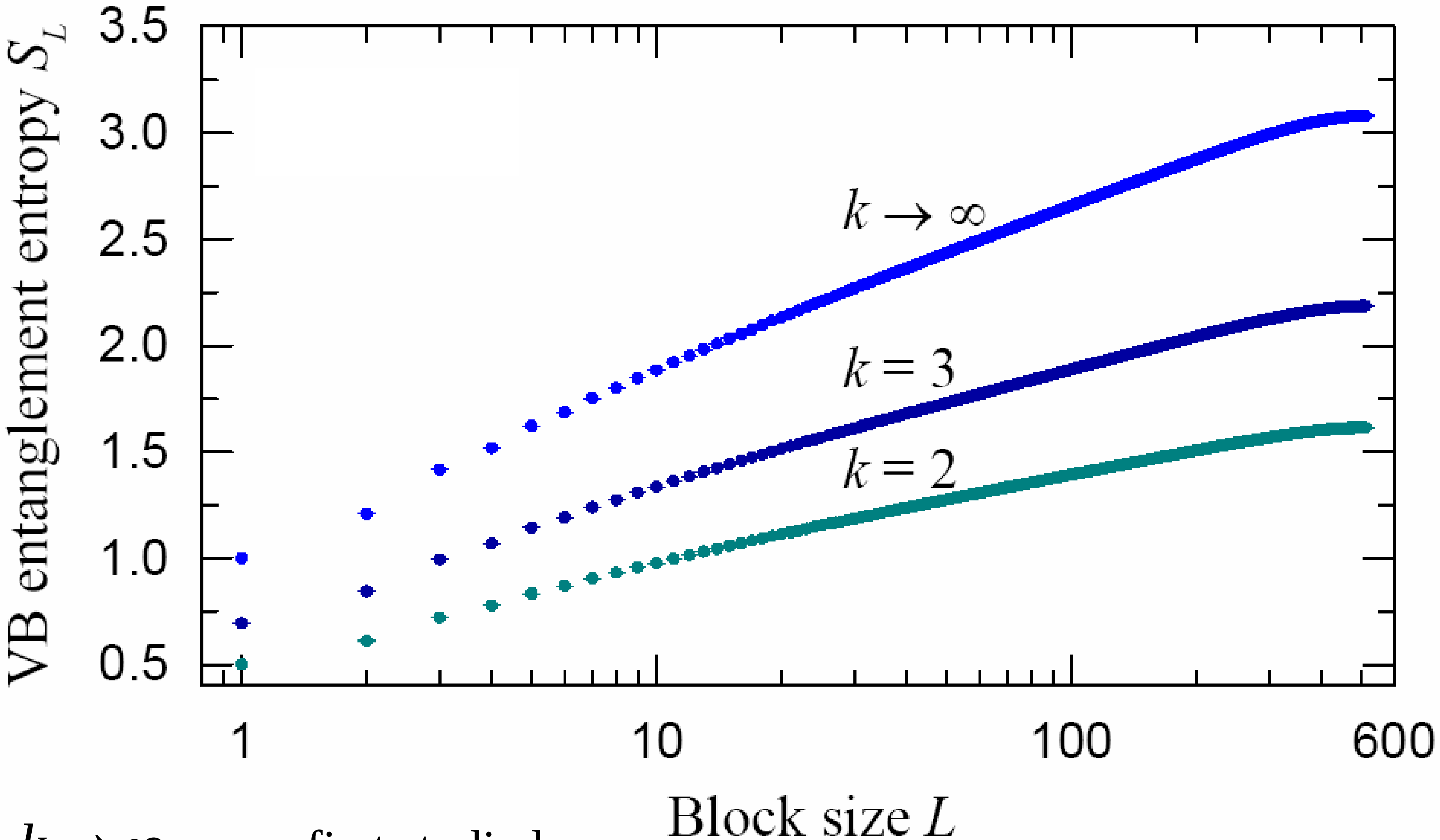
Valence-Bond Entanglement: Random Case



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H. Tran, NEB, in preparation

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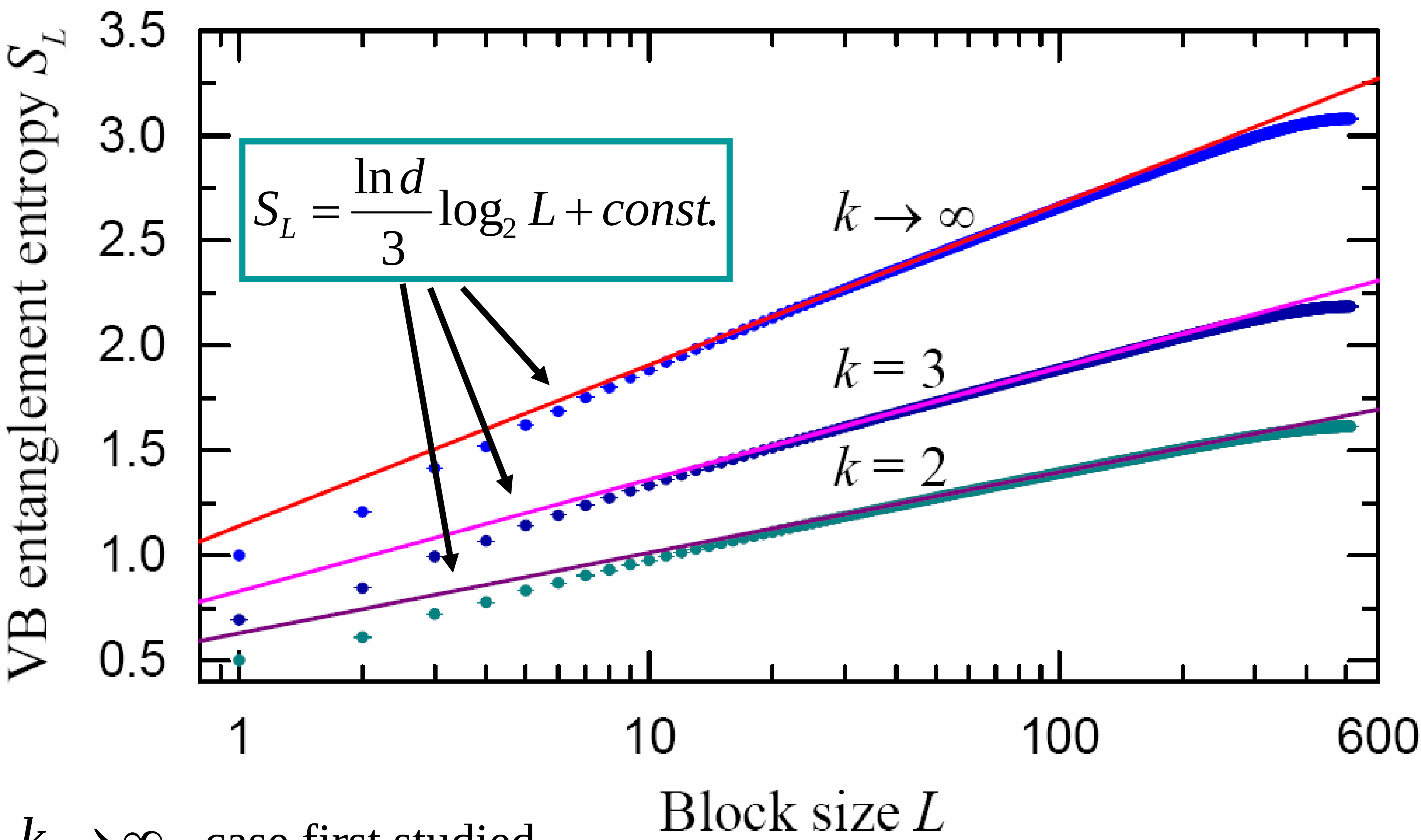

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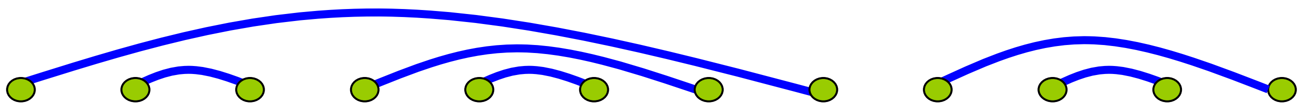
Conclusions

There is a close analogy between the properties of $SU(2)_k$ non-Abelian quasiparticles and ordinary spin-1/2 particles.

Chains of interacting non-Abelian particles can enter “random singlet phases,” analogous to those arising in random spin-1/2 chains.



Universal entanglement scaling.

$$S(L) \approx \frac{\ln d}{3} \log_2 L$$