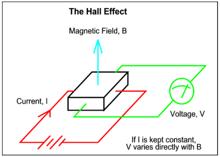
Highly entangled quantum matter

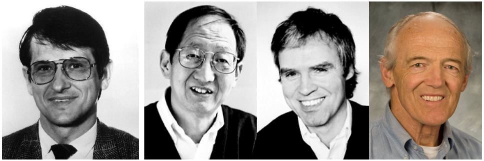
Xiao-Gang Wen, Perimeter, Nov, 2012

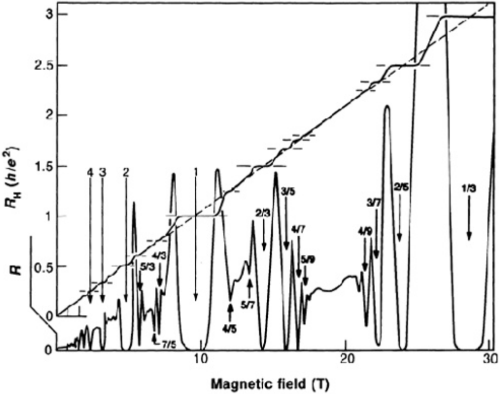
There are new phases beyond symmetry-breaking



$E_y = R_H j_x, \quad R_H = \frac{p}{q}\frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states** that all have the **same symmetry**.
- Different QH states can-not be described by symmetry breaking theory.
- We call the new order **topological order** Wen 89



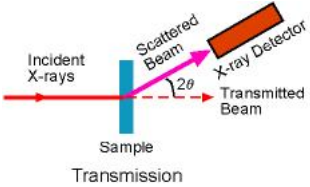


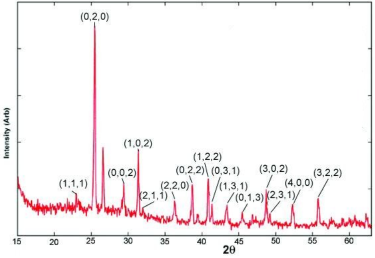
What is topological order?

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it

For example,

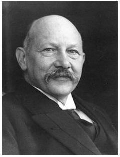
- crystal order is defined/probed by X-ray diffraction:

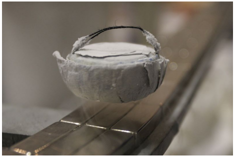




Symmetry-breaking orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order	???



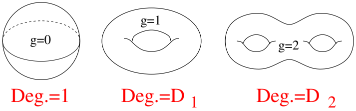


- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.

Topological orders through experiments (1990)

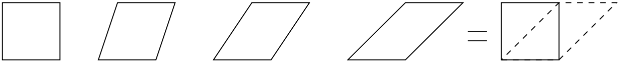
Topological order can be defined "experimentally" through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** D_g Wen 89



(2) **Non-Abelian geometric's phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation T: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$



- 90° rotation S: $|\Psi_\alpha\rangle \to |\Psi''_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

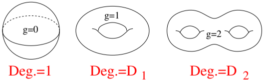
- T, S, define topological order "experimentally".
- T, S is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.

Symmetry-breaking/topological orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (Global dancing pattern)	Topological degeneracy, non-Abelian geometric phase

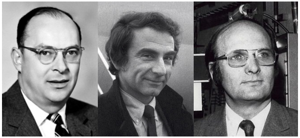
- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** T, S define a completely new class of order – **topologically order**.
- T, S determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

What is the microscopic picture of topological order?



represent an experimental definition of topological order.

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect → experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

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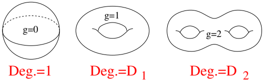

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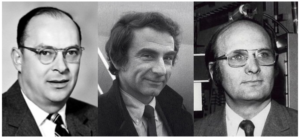

Bardeen-Cooper-Schrieffer 57

- It took 20 years to gain a microscopic understanding of topological order: **long-range entanglements** Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06,Levin-Wen 06

Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\rightarrow$ all systems belong to one trivial phase

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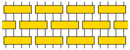
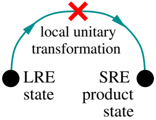
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



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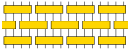
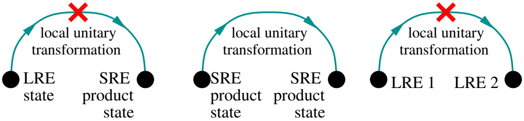
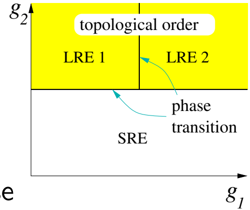
  $\to$ all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states** $\to$ many phases
  - There are **short range entangled (SRE) states** $\to$ one phase

$|\mathrm{LRE}\rangle \neq$  $|\text{product state}\rangle = |\mathrm{SRE}\rangle$

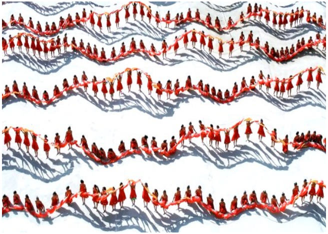




- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders**
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

Topological orders through pictures

FQH state



String liquid (spin liquid)

- **Global dance**:

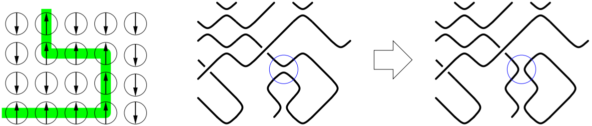
  All spins/particles dance following a local dancing "rules"

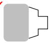
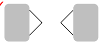
  → The spins/particles dance collectively

  → a global dancing pattern

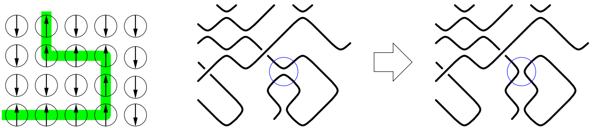
  → an entangled ground state wave function.

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=$ Φ_{str} () , Φ_{str} () $=$ Φ_{str} ()
  → Global dancing pattern Φ_{str} () $= 1$

Local dancing rule $\to$ global dancing pattern



- Local dancing rules of a string liquid:
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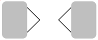
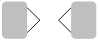
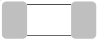
  (2) $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$, $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$
  $\to$ Global dancing pattern $\Phi_{\text{str}}\Big($$\Big) = 1$

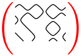
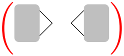
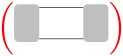
- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$, $\Phi_{\text{str}}\Big($$\Big) = -\Phi_{\text{str}}\Big($$\Big)$
  $\to$ Global dancing pattern $\Phi_{\text{str}}\Big($$\Big) = (-)^{\#\text{ of loops}}$

- Two string-net condensations $\to$ two topological orders Levin-Wen 05

Emergence of fractional spin/statistics (from the local dancing rules)

- Ends of string are topological defects in string liquids. They can carry fractional spins and fractional statistics

  Levin-Wen 05;Fidkowski-Freedman-Nayak-Walker-Wang 06

- Φ_{str}  $= 1$ string liquid Φ_{str}  $= \Phi_{\text{str}}$ 

  360° rotation:  $\rightarrow$  and  $=$  $\rightarrow$ 

   $+$  has a spin 0 mod 1.  $-$  has a spin $1/2$ mod 1.

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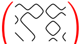
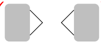
- $\Phi_{\text{str}}\Big($  $\Big) = 1$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

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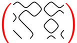
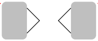
- $\Phi_{\text{str}}\Big($  $\Big) = (-)^{\#\text{ of loops}}$ string liquid $\Phi_{\text{str}}\Big($  $\Big) = -\Phi_{\text{str}}\Big($  $\Big)$

  360° rotation: $\to$ and $= -$ $\to -$

  $+\,i$ has a spin $-1/4$ mod 1. $-\,i$ has a spin $1/4$ mod 1.

Gapped phases w/ symmetry → SET and SPT phases

- **there are** LRE symmetric states → Symm. Enriched Topo. phases
  - 100s symm. spin liquid through the PSG of topo. excit. Wen 02
  - 8 trans. symm. enriched Z_2 topo. order in 2D, 256 in 3D Kou-Wen 09
  - $1000, 000$s symm. Z_2 spin liquid through $[\mathcal{H}^2(SG, Z_2)]^2 \times$ Hermele 12
  - Classify SET phases through $\mathcal{H}^3[SG \times GG, U(1)]$ Ran 12

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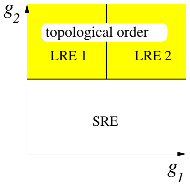
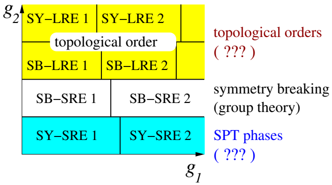
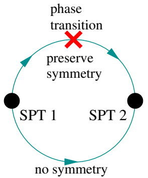
We may call them **symmetry protected trivial (SPT)** phase







- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

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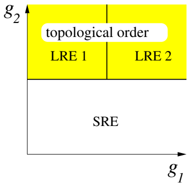
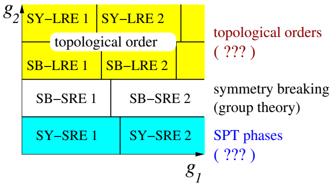
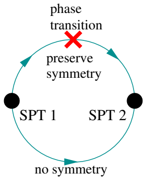
We may call them **symmetry protected trivial (SPT)** phase
or **symmetry protected topological (SPT)** phase







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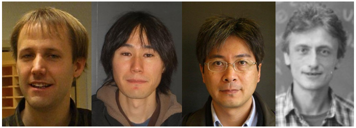
- 1 topo. ins. w/ $U(1) \times T$ symm. in 2D, Kane-Mele 05; Bernevig-Zhang 06
  15 in 3D Moore-Balents 07; Fu-Kane-Mele 07

Free fermion SPT phases: A K-theory

q	$\pi_0(R_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		no symmetry (p_x+ip_y, e.g., SrRu)	T only (^{3}He-B)
1	$\mathbb{Z}_2$	no symmetry (Majorana chain)	T only $((p_x+ip_y)\uparrow+(p_x-ip_y)\downarrow)$	T and Q (BiSb)
2	$\mathbb{Z}_2$	T only ((TMTSF)$_2$X)	T and Q (HgTe)	
3	0	T and Q		
4	$\mathbb{Z}$			
5	0			
6	0			
7	0			no symmetry

Kitaev 08

	TRS	PHS	SLS	d=1	d=2	d=3
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$



Schnyder-Ryu-Furusaki-Ludwig 08

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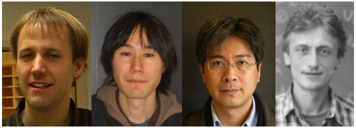
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$



Schnyder-Ryu-Furusaki-Ludwig 08

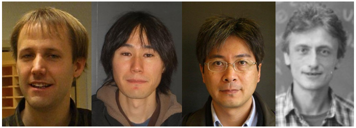
- How to include strong interactions $\rightarrow$ **Mission impossible**?

Free fermion SPT phases: A K-theory

q	$\neq_0(R_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		no symmetry (p_x+ip_y, e.g., SrRu)	T only (^{3}He-B)
1	$\mathbb{Z}_2$	no symmetry (Majorana chain)	T only $((p_x+ip_y)\uparrow+(p_x-ip_y)\downarrow)$	T and Q (BiSb)
2	$\mathbb{Z}_2$	T only ($(TMTSF)_2X$)	T and Q (HgTe)	
3	0	T and Q		
4	$\mathbb{Z}$			
5	0			
6	0			
7	0			no symmetry

Kitaev 08

	TRS	PHS	SLS	d=1	d=2	d=3
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$

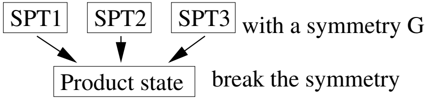


Schnyder-Ryu-Furusaki-Ludwig 08

- How to include strong interactions → **Mission impossible**?
- SPT phases are 'trivial' (short-range entangled) → **Mission possible**

Group theory classifies 230 crystal orders. What classifies SPT orders?

- *Symmetry protected topological (SPT) phases* are gapped quantum phases with certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry.



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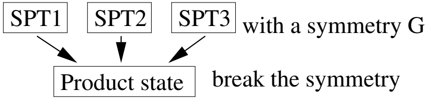


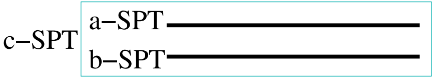
- **A classification of (all?) SPT phase**: Chen-Gu-Liu-Wen 11
  For a system in d spatial dimension with an on-site symmetry G, its SPT phases that do not break the symmetry G are classified by the elements in $\mathcal{H}^{d+1}[G, U_T(1)]$ – the $d+1$ cohomology class of the symmetry group G with G-module $U_T(1)$ as coefficient.

- **Characteristic properties of the SPT phases**: Chen-Gu-Liu-Wen 11
  A SPT phase characterized by a non-trivial element in $\mathcal{H}^{d+1}[G, U_T(1)]$ has gapless/degenerate boundary states if the symmetry is not broken on the boundary.

What is d-cohomology class $\mathcal{H}^d[G, U_T(1)]$ of a group G

- $\mathcal{H}^d[G, U_T(1)]$ is set: $\mathcal{H}^d[G, U_T(1)] = \{a, b, c, ...\}$

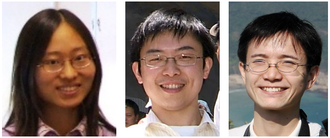
- The set $\mathcal{H}^d[G, U_T(1)]$ form an Abelian group (with an addition operation): $a``+"b = c, \quad a, b, c \in \mathcal{H}^d[G, U_T(1)]$
  - Stacking a-SPT state and b-SPT state give us a c-SPT state.



- The group $\mathcal{H}^d[G, U_T(1)]$ contain an identity element $\mathbf{0}$ under the addition operation: $\mathbf{0}``+"a = a, \quad \mathbf{0}, a \in \mathcal{H}^d[G, U_T(1)]$
  - $\mathbf{0} \to$ trivial SPT state
  - $a \neq \mathbf{0} \to$ non-trivial SPT states

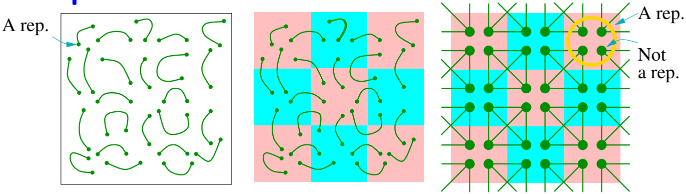
Interacting bosonic SPT phase: A group-cohomolgy theory

Chen-Liu-Wen 11, Chen-Gu-Liu-Wen 11



For any symmetry group G
and in any dimensions d
Two key observations:

- **Short-range-entangled states have a simple canonical form**:



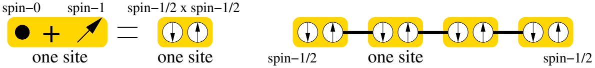
after we treat each block as an effective site.

- *Each effective site has several independent degrees of freedoms entangled with its neighbors.*
- *The combined degrees of freedoms on a site form a rep. of G*

- **Each degree of freedoms on the effective site may not form a representation of G, but the whole state is still inv. under G**

Non-trivial short-range entangled states w/ symmetry

- **Haldane phase w/ $SO(3)$ symm.**: spin-1/2 is not a rep. of $SO(3)$



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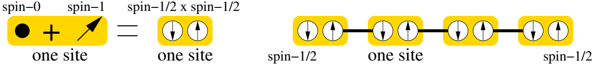
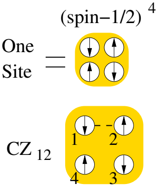
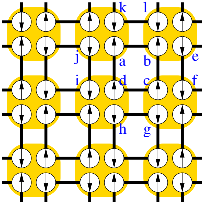


- **2D SPT phase w/ Z_2 symm.**:

Chen-Liu-Wen 2011

- Physical states on each site: $(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$
- The ground state wave function: $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$





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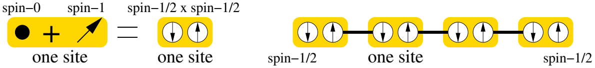

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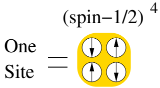

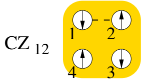

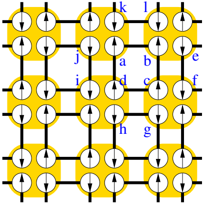

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- The ground state wave function:
  $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$

- The on-site Z_2 symmetry: (acting on each site $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$):
  $U_{CZX} = U_{CZ} U_X, \quad U_X = X_1 X_2 X_3 X_4, \quad U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}$
  $CZ : |\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \to -|\downarrow\downarrow\rangle$

Non-trivial short-range entangled states w/ symmetry

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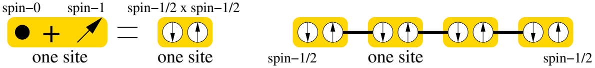

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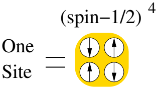

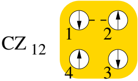

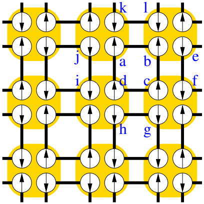

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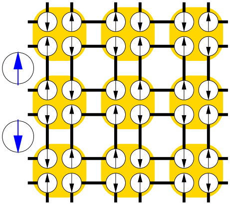
$CZ : |\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \to -|\downarrow\downarrow\rangle$
- Z_2 symm. Hamiltonian $H = \sum_\square H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,
$X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$, $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

Edge excitations for the 2D Z_2 SPT state

- Bulk Hamiltonian $H = \sum_{\square} H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij}, P_{kl}$,
  $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$, $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

- Edge excitations: *gapless or break the $\mathbb{Z}_2$ symmetry, robust against any perturbations that do not break the $\mathbb{Z}_2$ symmetry.*



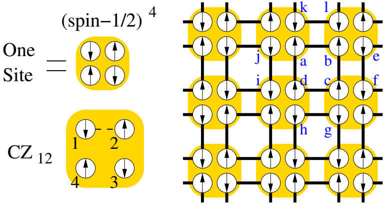
- Edge effective spin $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$.
- Edge effective Z_2 symmetry : $\exp\Big(\sum_i \frac{1}{4}(\tilde{Z}_i \tilde{Z}_{i+1} - 1)\Big) \prod_i \tilde{X}_i$
  which cannot be written as $U_{Z_2} = \prod_i O_i$, *such as* $U_{Z_2} = \prod \tilde{X}_i$.
  Not an on-site symmetry!
- Edge effective Hamiltonian ($c = 1$ gapless if the Z_2 is not broken)

$$H_{\text{edge}} = -J \sum \tilde{Z}_i \tilde{Z}_{i+1} + B_x \sum [\tilde{X}_i + \tilde{Z}_{i-1} \tilde{X}_i \tilde{Z}_{i+1}]$$
$$+ B_y \sum [\tilde{Y}_i - \tilde{Z}_{i-1} \tilde{Y}_i \tilde{Z}_{i+1}]$$

SPT states for any symmetry in any dimensions

- Generic SPT state: $(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle \to \sum_{g\in G} |gggg\rangle)$

$$|\text{SPT state}\rangle = \otimes_{\text{all squares}} \sum_{g\in G} |gggg\rangle$$



- Generic twisted symmetry transformations

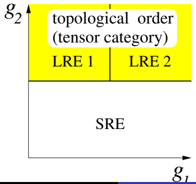
$$|g_1, g_2, g_3, g_4\rangle \to \eta(g_1, g_2, g_3, g_4)|gg_1, gg_2, gg_3, gg_4\rangle$$

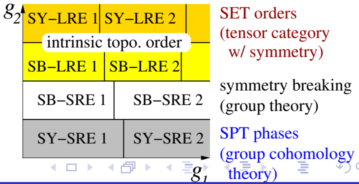
where the twisting phase factor $\eta(g_1, g_2, g_3, g_4)$ correspond to cocycles in $\mathcal{H}^{d+1}[G, U_T(1)]$.

Bosonic SPT phases in any dim. and for any symmetry

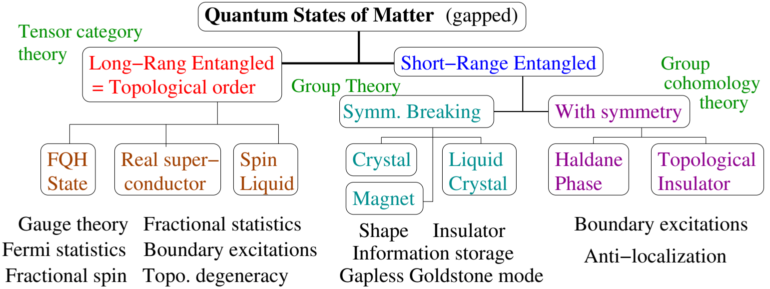
Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U(1) \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U(1) \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times$ trans	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SO(3)$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$

Table of $\mathcal{H}^{d+1}[G, U_T(1)]$
"Z_2^T": time reversal,
"trans": translation,
others: on-site symm.
$0 \to$ only trivial phase.
$(\mathbb{Z}_2) \to$ free fermion result





Highly entangled quantum matter:
A new chapter of condensed matter physics



- Group theory → Symmetry breaking order →
  shape, superfluid, phonon, magnets, magnon, liquid crystals, ...
- Tensor category theory → Topological order →
  FQH effect, anyons, fermions, fractional charge/spin, spin liquid, photon, perfect conducting edges, ...
- Group cohomology theory → SPT order →
  symmetry protected boundary excitations, ...