

IUIC/PI workshop:  
**Topology, Entanglement and Strong Correlations  
in Condensed Matter**

*Towards a complete characterization of  
Emergent Topological Order  
from a microscopic Hamiltonian*

Guifre Vidal  
Perimeter Institute

Based on *Lukasz Cincio, G. V., arXiv:1208.2623*

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in Condensed Matter**

*Towards a complete characterization of  
Emergent Topological Order  
from a microscopic Hamiltonian*

Collaboration with  
Lukasz Cincio  
(Perimeter)



Perimeter Institute

Based on *Lukasz Cincio, G. V., arXiv:1208.2623*

$H$



## emergent anyon model

microscopic  
Hamiltonian

- number of topological fluxes/anyon types

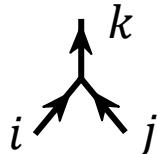
[toric code:  $\mathbb{I}, e, m, \varepsilon$ ]

[Ising:  $\mathbb{I}, \sigma, \varepsilon$ ]

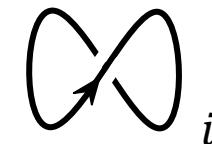
- quantum dimensions

$$d_i \quad D = \sqrt{\sum_i (d_i)^2}$$

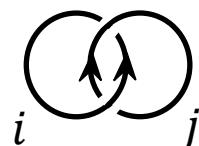
- fusion rules  $N_{ij}^k$



- topological spin  $\theta_i$   
topological central charge  $C$



- mutual statistics  $S_{ij}$



...



(if gapless edge state)  
chiral CFT

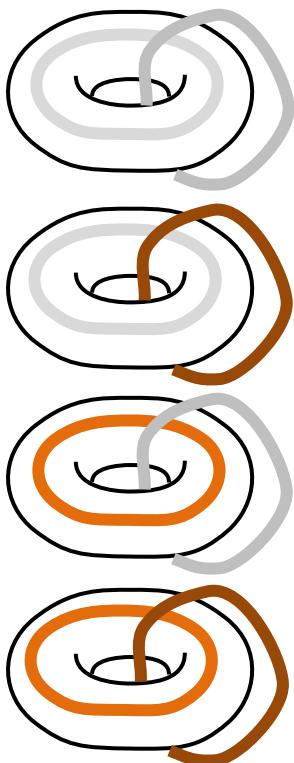
# Clarification (1)

## Kitaev's toric code (quantum double of $Z_2$ )

Ground subspace on the torus

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

One possible basis:



$$\mathbb{I} \times \mathbb{I}$$

$$|00\rangle$$

$$\mathbb{I} \times m$$

$$|01\rangle$$

$$m \times \mathbb{I}$$

$$|10\rangle$$

$$m \times m$$

$$|11\rangle$$

Another basis:

$$\mathbb{I}$$

$$|0+\rangle$$

$$e$$

$$|0-\rangle$$

$$m$$

$$|1+\rangle$$

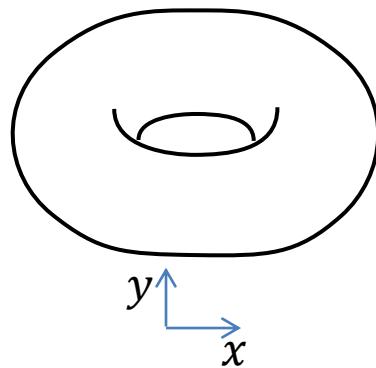
$$\varepsilon = em$$

$$|1-\rangle$$

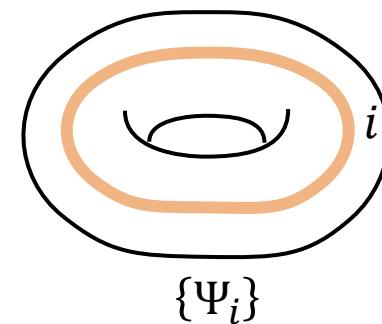
## Clarification (2)

- complete set of ground states of a lattice Hamiltonian  $H$

A) on a torus



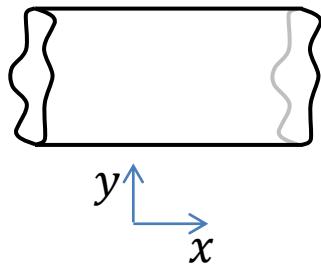
$$L_x, L_y \gg \xi$$



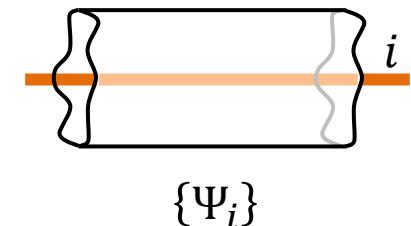
fact: each ground state has a well-defined anyon flux in x-direction

example: toric code  $i = \mathbb{I}, e, m, \varepsilon$

B) on an infinite cylinder (*with no boundaries*)



$$L_x = \infty; \quad L_y \gg \xi$$



fact: each ‘ground state’ has a well-defined anyon flux in x-direction

example: toric code  $i = \mathbb{I}, e, m, \varepsilon$

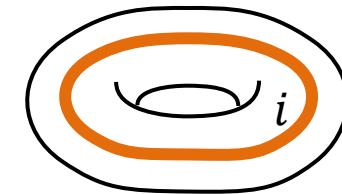
X.-G. Wen, 1989

# Background

Infinite cylinder



Finite torus



Entanglement spectrum

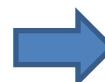


spectrum of  
gapless edge state  
(chiral CFT)

$$H_i^{(\text{boundary})}$$

H. Li, F. D. M. Haldane, PRL 2008  
X.-L. Qi, H. Katsura, A. W. W. Ludwig, PRL 2012

Topological  
entanglement entropy

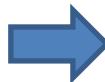


quantum dimensions

$$\frac{d_i}{D} \qquad D = \sqrt{\sum_i (d_i)^2}$$

A. Kitaev, J. Preskill, PRL 2006, M. Levin, X.-G. Wen, PRL 2006  
S. Dong, E. Fradkin, R. Leigh, S. Nowling, JHEP 2008

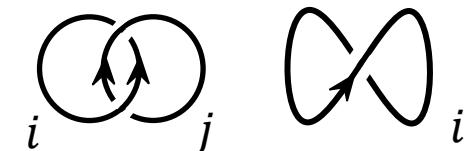
Modular transformations



topological  $S, U$  matrices

$$V_{ij} = \langle \Psi_i^{\text{tor}} | R_{\pi/3} | \Psi_j^{\text{tor}} \rangle$$

$$V = DUS^{-1}D^\dagger$$



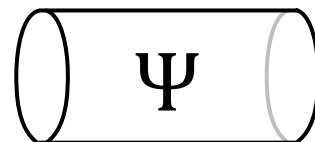
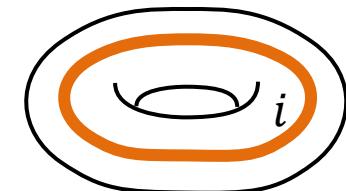
Y. Zhang, T. Grover, A. Turner, M. Oshikawa, A. Vishwanath, PRB 2012

# Background

Infinite cylinder



Finite torus



ground state  
on finite cylinder

2D DMRG

S. Yan, D. A. Huse, S. R. White, Science 2011



$$D = \sqrt{\sum_i (d_i)^2}$$

H.-C. Jiang, H. Yao, L. Balents, PRB 2012

H.-C. Jiang, Z. Wang, L. Balents, arXiv:1205.4289

S. Depenbrock, I. P. McCulloch, U. Schollwoeck, PRL 2012

# OUTLINE

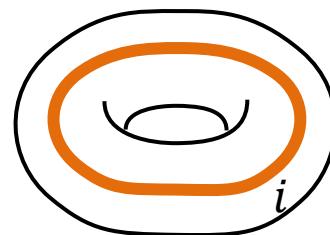
## 1) GROUND STATES

Infinite cylinder



- edge spectrum
  - quantum dimensions
  - chiral CFT

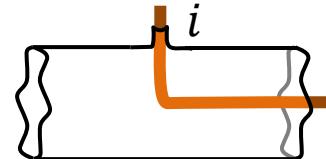
Finite torus



- S matrix
  - mutual statistics
  - quantum dimensions
  - fusion rules
- U matrix
  - central charge
  - topological spins

## 2) QUASIPARTICLE EXCITATIONS

Infinite cylinder

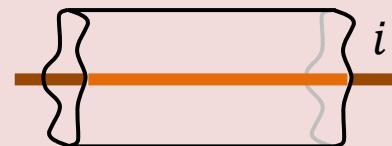


- integer excitations
- fractionalized excitations

# OUTLINE

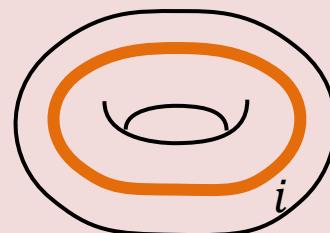
## 1) GROUND STATES

Infinite cylinder



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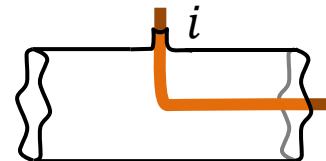
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  - topological spins

## 2) QUASIPARTICLE EXCITATIONS

Infinite cylinder



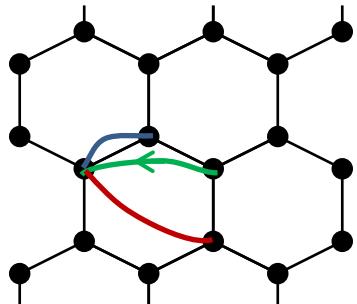
- integer excitations
- fractionalized excitations

# LATTICE MODELS

## Haldane

(hardcore bosons on honeycomb)

F.D.M. Haldane, PRL 1988



$$t = 1$$

$$t' = 0.6$$

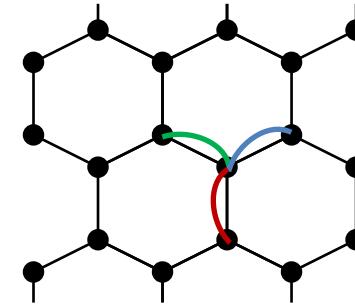
$$\phi = 0.4\pi$$

$$t'' = -0.58$$

## Kitaev Honeycomb

(non-Abelian phase with magnetic field)

A. Kitaev , Annals of Physics 2006



$$h=0.01$$

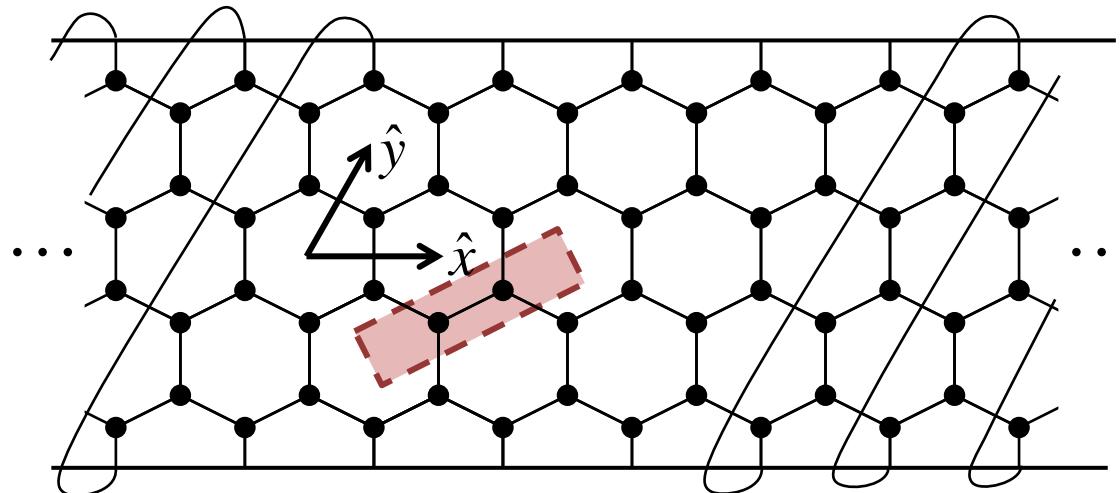
$$H = -t \sum_{\langle rr' \rangle} b_r^\dagger b_{r'} - t' \sum_{\langle\langle rr' \rangle\rangle} b_r^\dagger b_{r'} e^{i\phi_{rr'}} - t'' \sum_{\langle\langle\langle rr' \rangle\rangle\rangle} b_r^\dagger b_{r'}$$

Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D.N. Sheng, PRL 2011

$$H = \sum_{\langle rr' \rangle_x} \sigma^x_r \sigma^x_{r'} + \sum_{\langle rr' \rangle_y} \sigma^y_r \sigma^y_{r'} + \sum_{\langle rr' \rangle_z} \sigma^z_r \sigma^z_{r'}$$

$$+ h \sum_r (\sigma^x_r + \sigma^y_r + \sigma^z_r)$$

# VARIATIONAL WAVEFUNCTION



$$L_x = \infty$$

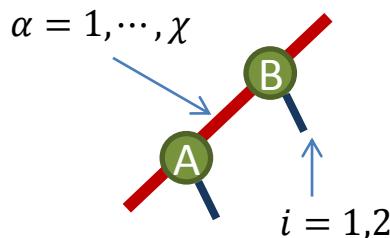
## MPS / 2D DMRG

(Matrix Product State)

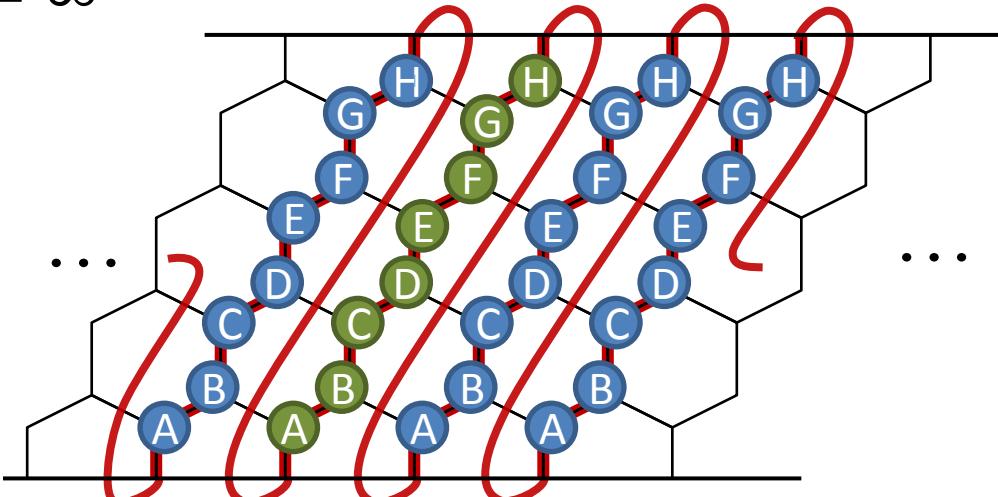
S. White, PRL 1992

S. Yan, D. A. Huse, S. R. White, Science 2011

H.-C. Jiang, H. Yao, L. Balents, PRB 2012



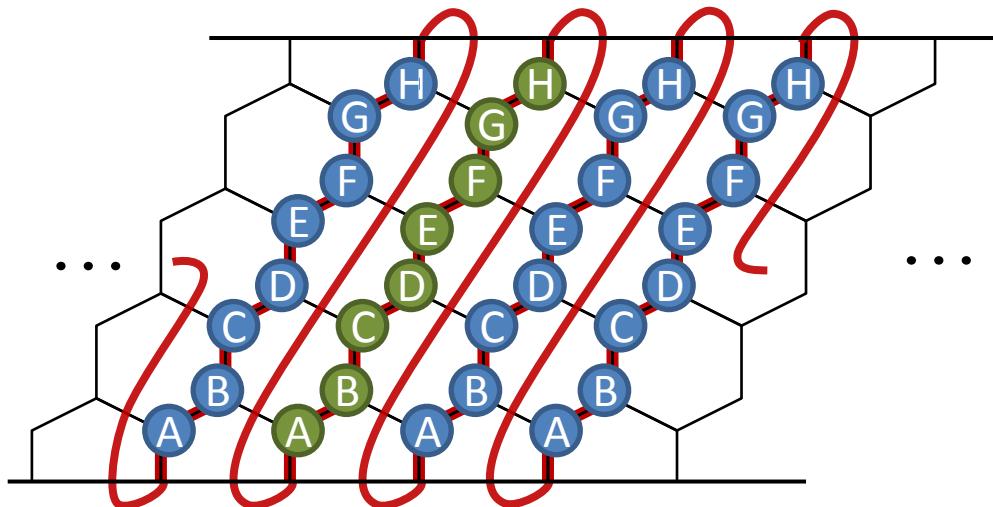
Computational cost



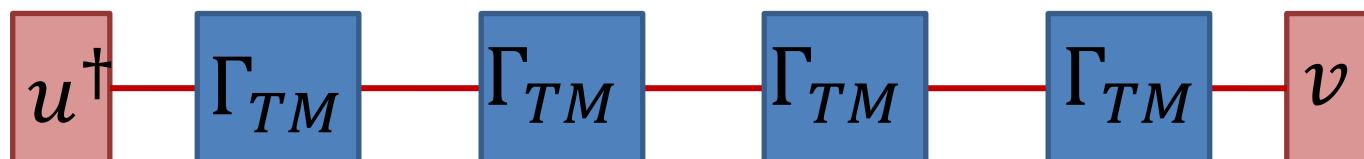
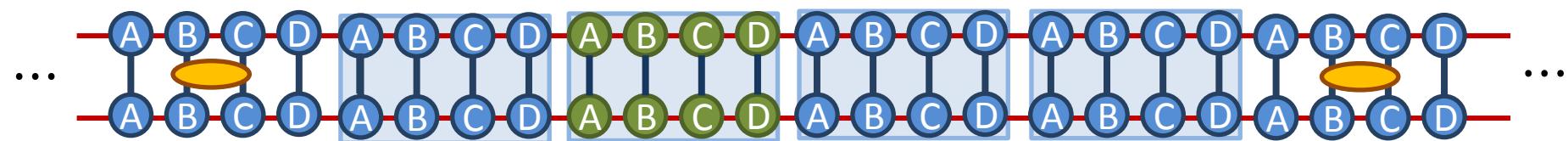
$$O(\chi^3) \sim e^{L_y}$$

$$L_y \gg \xi$$

# CORRELATION LENGTH



$$\langle \Psi | o(0,0) o(x,y) | \Psi \rangle =$$

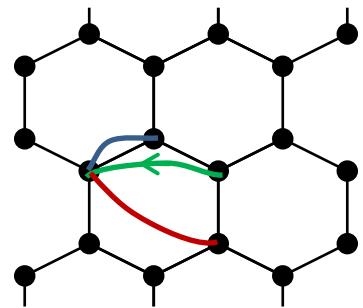


$$\approx \lambda^x = e^{-x/\xi_{TM}}$$

$$\xi_{TM} \stackrel{\text{def}}{=} -\frac{1}{\log(\lambda)}$$

## Haldane model (hardcore bosons)

F.D.M. Haldane, PRL 1988



$$t = 1$$

$$t' = 0.6$$

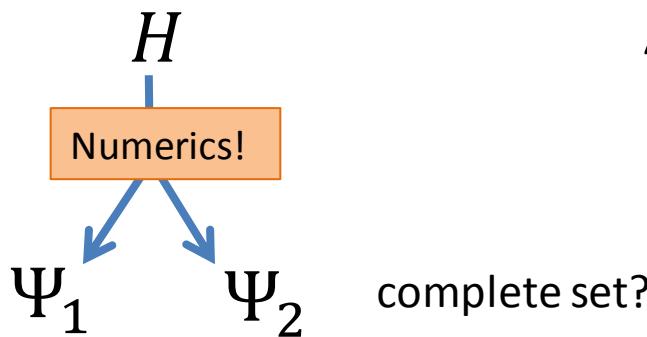
$$\phi = 0.4\pi$$

$$t'' = -0.58$$

$$H = -t \sum_{\langle rr' \rangle} b_r^\dagger b_{r'} - t' \sum_{\langle\langle rr' \rangle\rangle} b_r^\dagger b_{r'} e^{i\phi_{rr'}} - t'' \sum_{\langle\langle rr' \rangle\rangle} b_r^\dagger b_{r'}$$

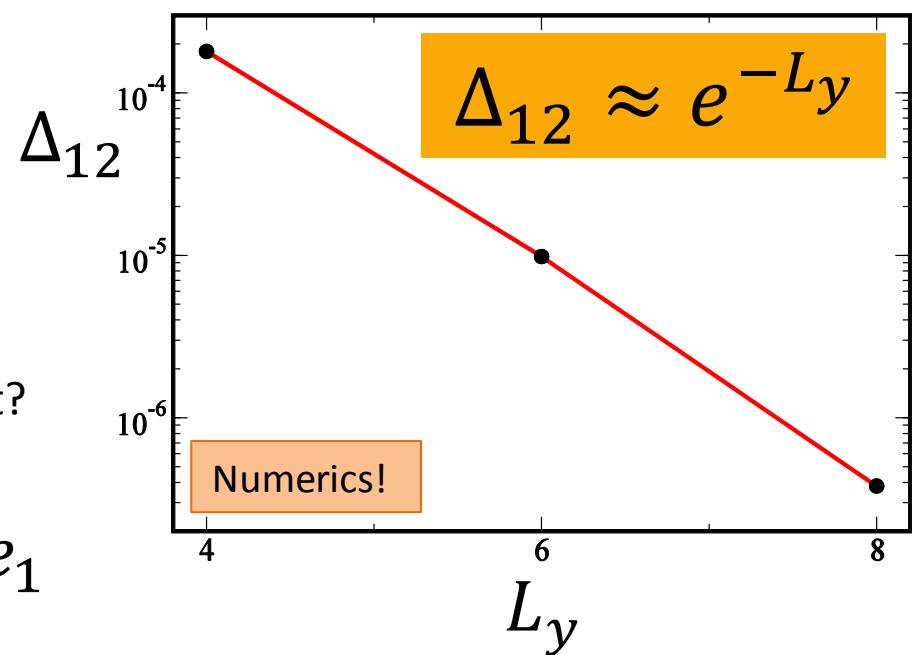
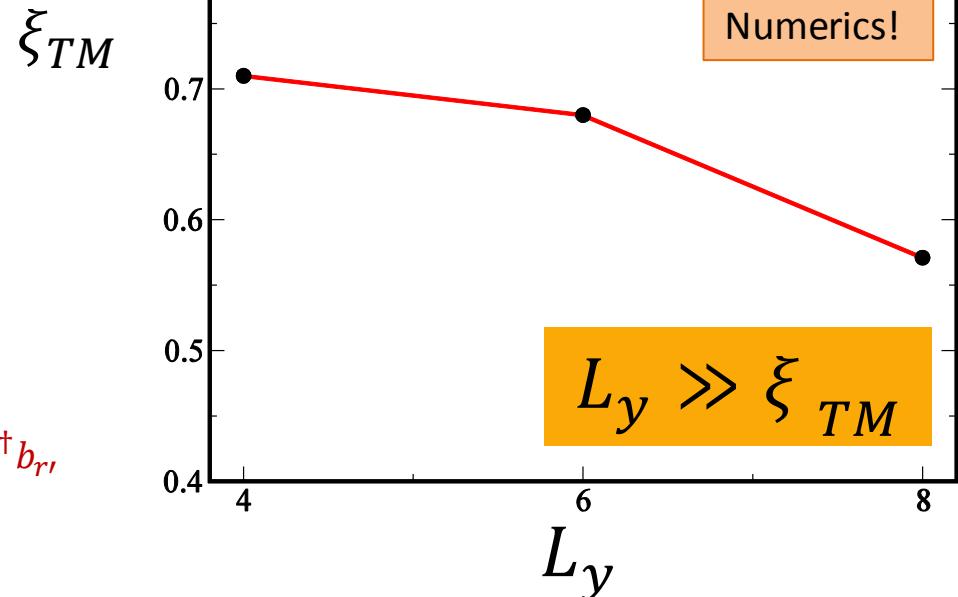
Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D.N. Sheng, PRL 2011  
**(flat band!)**

We find 2 ‘ground states’:



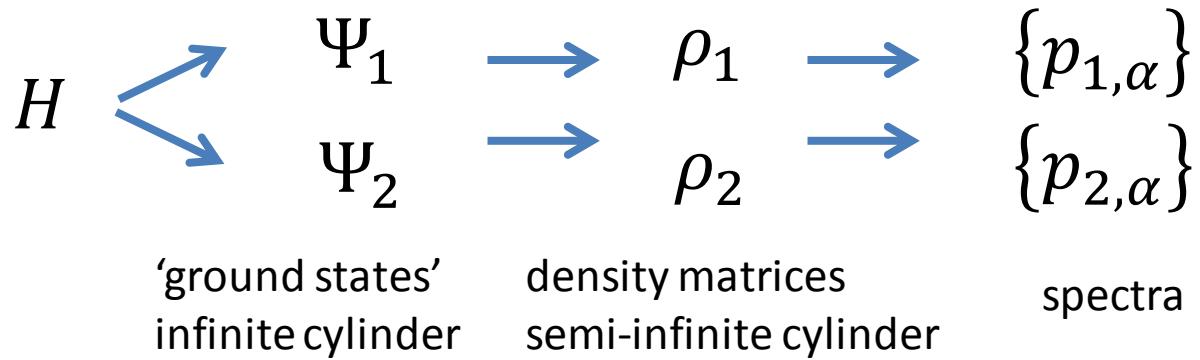
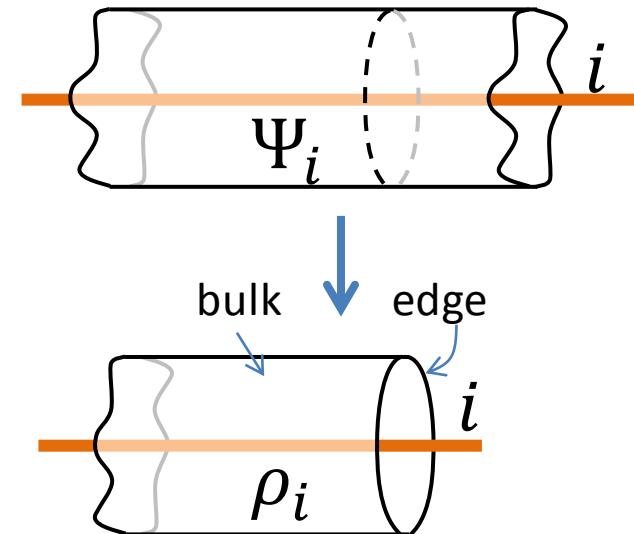
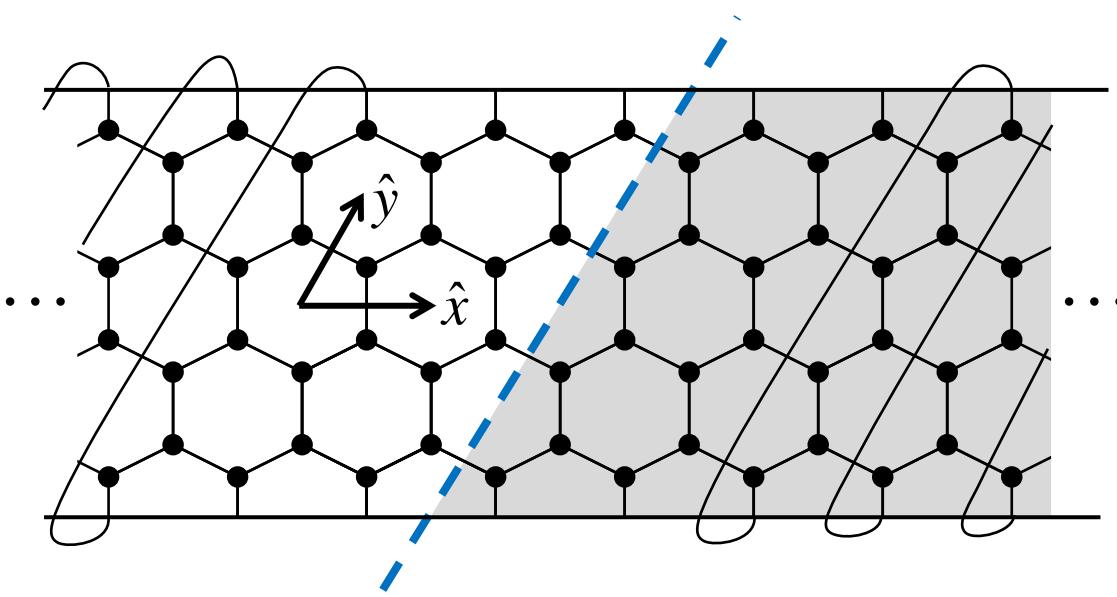
$$e_i \stackrel{\text{def}}{=} \frac{E_i}{L_x L_y}$$

$$\Delta_{12} \stackrel{\text{def}}{=} e_2 - e_1$$



Haldane model  
(hardcore bosons)

ENTANGLEMENT SPECTRUM (I)



$$\rho_i |p_{i,\alpha}\rangle = p_{i,\alpha} |p_{i,\alpha}\rangle$$

Haldane model  
(hardcore bosons)

$$\{p_{1,\alpha}\}, \{p_{2,\alpha}\} \rightarrow S(\rho_1), S(\rho_2)$$

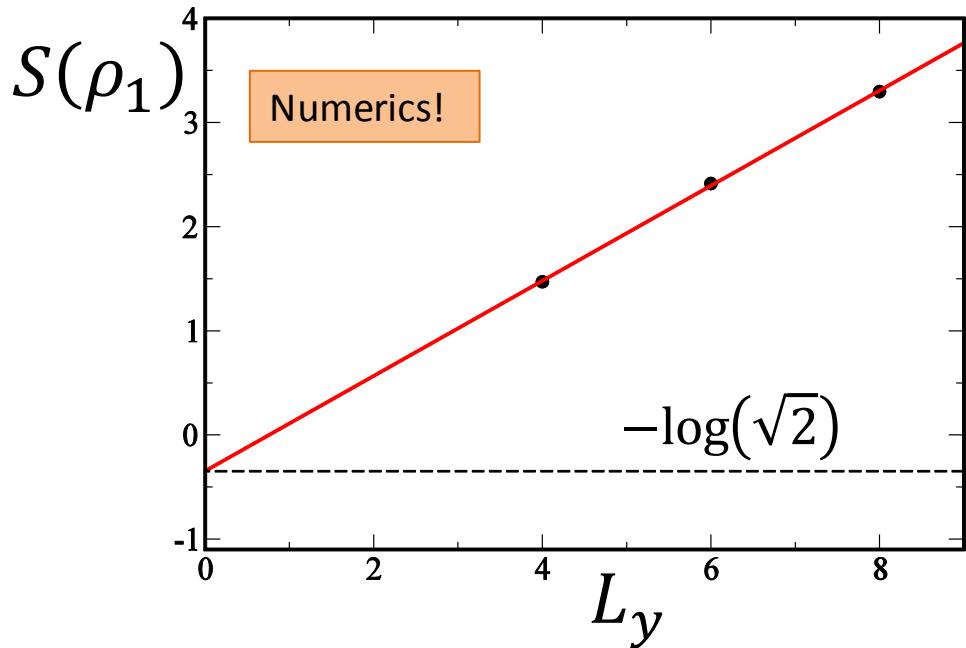
spectrum

Scaling of entanglement entropy

A. Kitaev, J. Preskill, PRL 2006

M. Levin, X.-G. Wen, PRL 2006

S. Dong, E. Fradkin, R. Leigh, S. Nowling, JHEP 2008



Region with flux  $i$

$$S_L = aL - \log\left(\frac{D}{d_i}\right)$$

\*For one ground state in large finite cylinder, H.-C. Jiang, H. Yao, L. Balents, PRB 2012,  
H.-C. Jiang, Z. Wang, L. Balents, arXiv:1205.4289

Numerics!

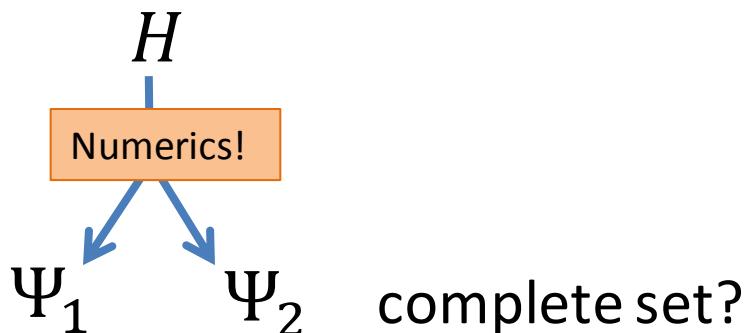
$$\frac{d_1}{D} = 0.7079 \approx \frac{1}{\sqrt{2}} \text{ (0.1%)}$$

$$S(\rho_1) - S(\rho_2) = \log\left(\frac{d_1}{d_2}\right)$$

Numerics!

$$d_1/d_2 = 1.005$$

We found 2 ‘ground states’:



Numerics!

$$\left(\frac{d_1}{D}\right)^2 + \left(\frac{d_2}{D}\right)^2 = 1.007$$

⇒ complete set

$$D \stackrel{\text{def}}{=} \sqrt{\sum_i (d_i)^2}$$
$$\Downarrow$$
$$\sum_i \left(\frac{d_i}{D}\right)^2 = 1$$

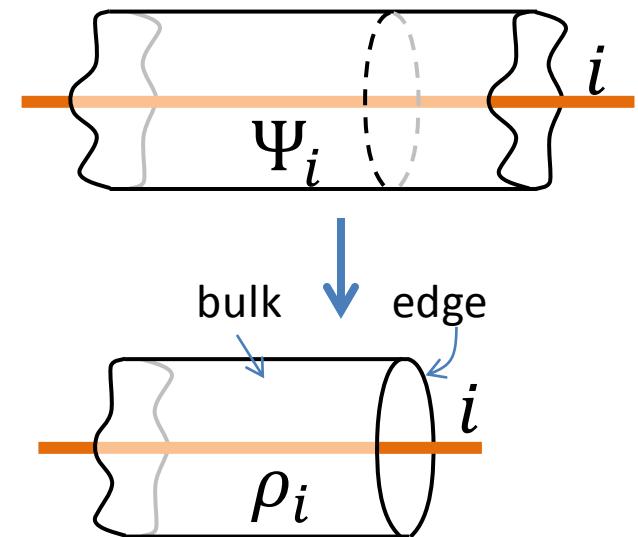
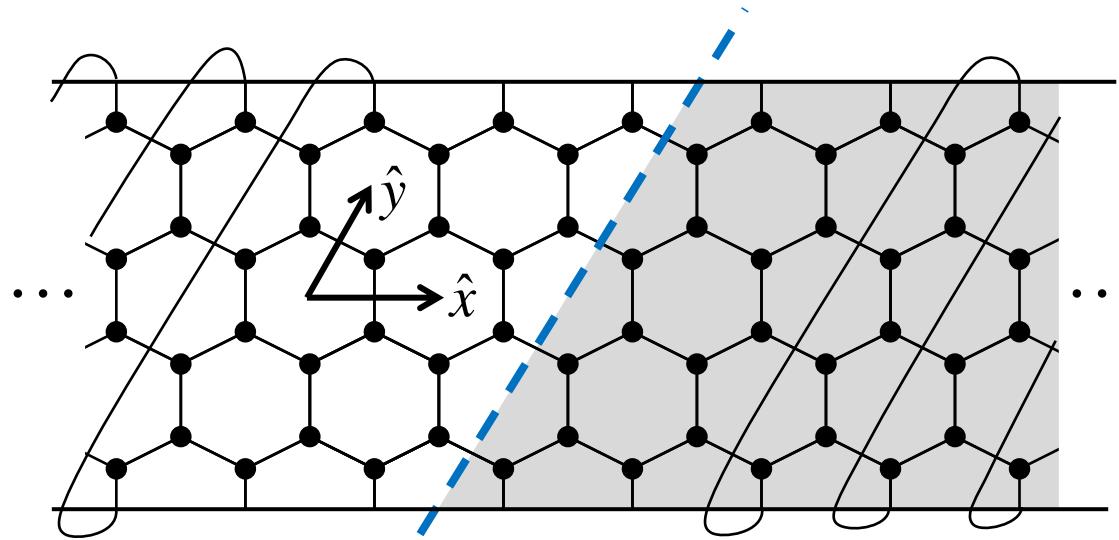
Any anyon model has identity  $i = \mathbb{I}$ , with quantum dimension  $d_{\mathbb{I}} = 1$

$$d_1 = 1, \quad \Rightarrow \quad d_2 = 1.005 \approx 1, \quad D = 1.413 \approx \sqrt{2} \quad (0.1\%),$$

Numerics!

Haldane model  
(hardcore bosons)

ENTANGLEMENT SPECTRUM (II)



$$H \begin{array}{c} \swarrow \\ \Psi_1 \\ \searrow \end{array} \begin{array}{c} \rightarrow \\ \Psi_2 \end{array} \begin{array}{c} \rightarrow \\ \rho_1 \\ \rightarrow \\ \rho_2 \end{array} \begin{array}{c} \{p_{1,\alpha}; k_{1,\alpha}\} \\ \{p_{2,\alpha}; k_{2,\alpha}\} \end{array}$$

'ground states'  
infinite cylinder

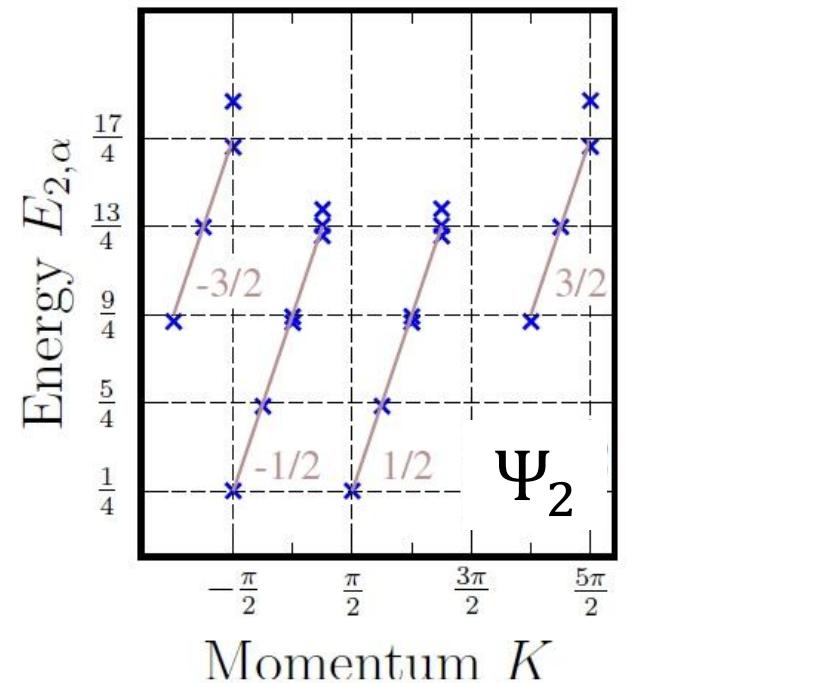
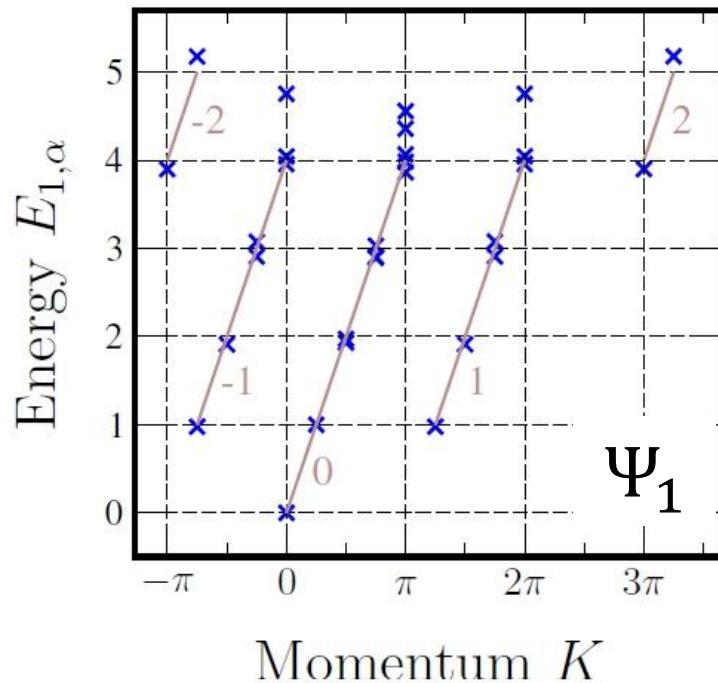
spectra  
entanglement energies

$$\rho_i |p_{i,\alpha}; k_{i,\alpha}\rangle = p_{i,\alpha} |p_{i,\alpha}; k_{i,\alpha}\rangle$$

$$E_i \stackrel{\text{def}}{=} -\log(p_{i,\alpha})$$

$$T_{y1} |p_{i,\alpha}; k_{i,\alpha}\rangle = e^{-i \frac{2\pi}{L_y} \textcolor{red}{k}_{i,\alpha}} |p_{i,\alpha}; k_{i,\alpha}\rangle$$

momentum in y-direction



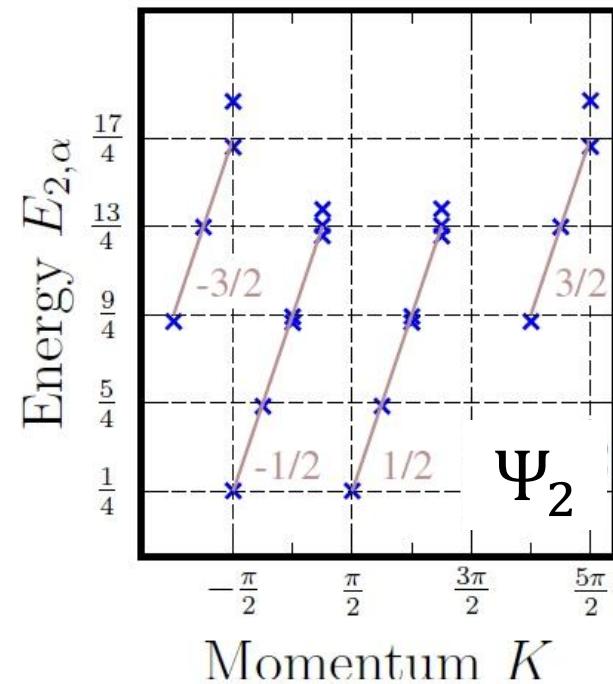
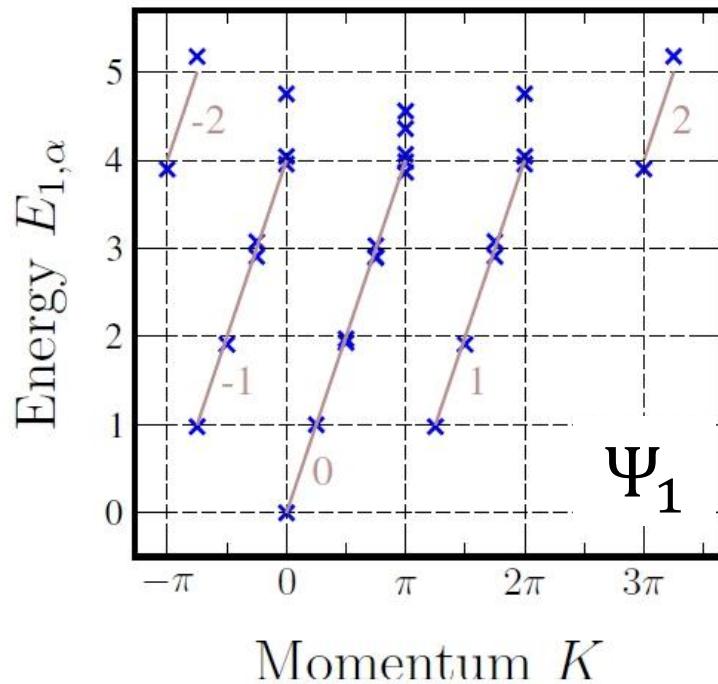
- Spectrum organized as multiplets of emergent SU(2) [lattice model is only U(1) symmetric]

$\Psi_1 \quad m_z = \dots -2, -1, 0, 1, 2 \dots$   
integer irreps  $s = 0, 1, 2, \dots$

$\Psi_2 \quad m_z = \dots -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$   
integer irreps  $s = 0, 1, 2, \dots$

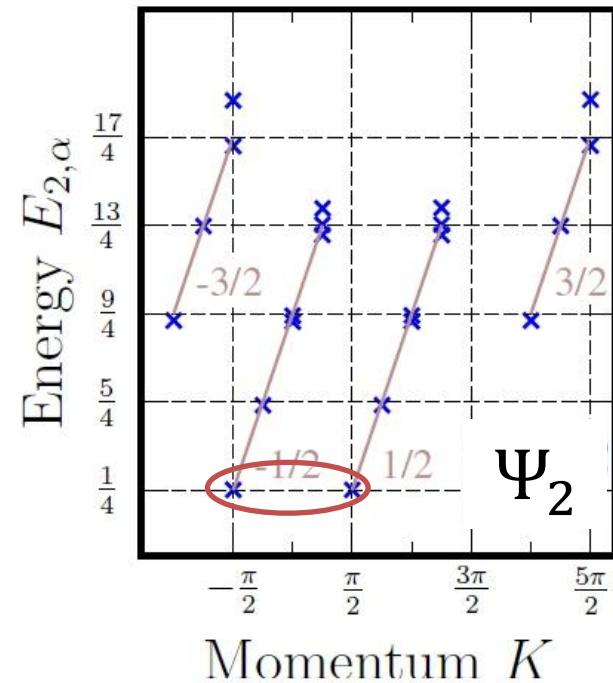
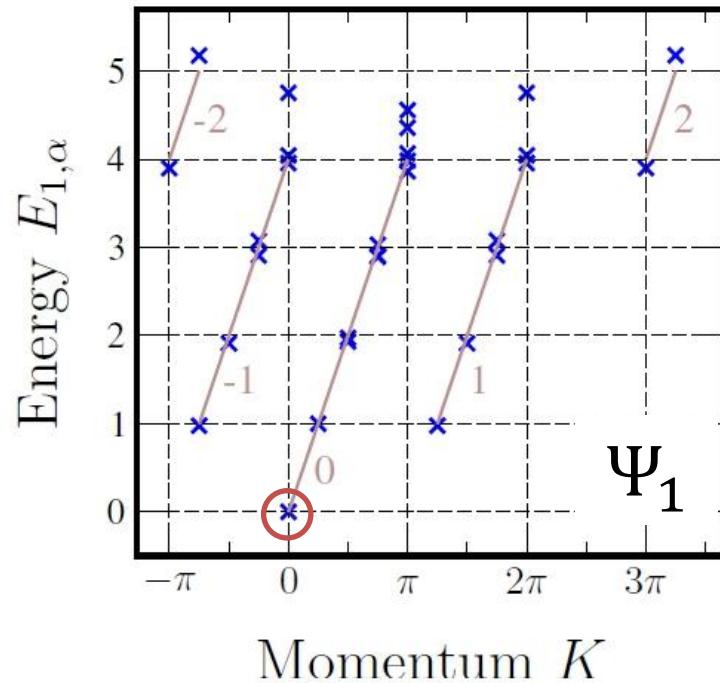
- Degeneracy pattern:  $\{1, 1, 2, 3, 5, \dots\}$

Xiao-Gang: “bosonic Gaussian theory”



$L_0$	$m$					$su(2)$ decomposition
	-2	-1	0	1	2	
0			1			(0)
1		1	1	1		(2)
2		1	2	1		(2)+(0)
3		2	3	2		2(2)+(0)
4	1	3	5	3	1	(4)+2(2)+2(0)
5	1	5	7	5	1	(4)+4(2)+2(0)
6	2	7	11	7	2	2(4)+5(2)+4(0)

$L_0$	$m$						$su(2)$ decomposition
	-2	-1	0	1	2	3	
$\frac{1}{4}$			1	1			(1)
$\frac{5}{4}$			1	1			(1)
$\frac{9}{4}$			1	2	2	1	(3)+(1)
$\frac{13}{4}$			1	3	3	1	(3)+2(1)
$\frac{17}{4}$			2	5	5	2	2(3)+3(1)
$\frac{21}{4}$			3	7	7	3	3(3)+4(1)
$\frac{25}{4}$	1	5	11	11	5	1	(5)+4(3)+6(1)



chiral  $SU(2)_1$  Wess-Zumino-Witten CFT

$\Psi_i$  primary field + tower of (Virasoro and Kac-Moody) descendants

$\Psi_1$  identity  $I$ ,  
 $SU(2)$  singlet



$\Psi_{\text{II}}$  identity

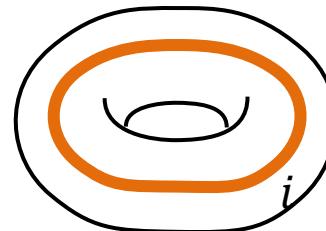
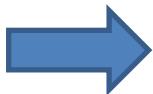
$\Psi_2$  chiral vertex operator  $e^{i\varphi/\sqrt{2}}$ ,  
 $SU(2)$  doublet



$\Psi_S$  semion



infinite cylinder



finite torus

complete set  
of 'ground states'

$\Psi_{\mathbb{I}}$

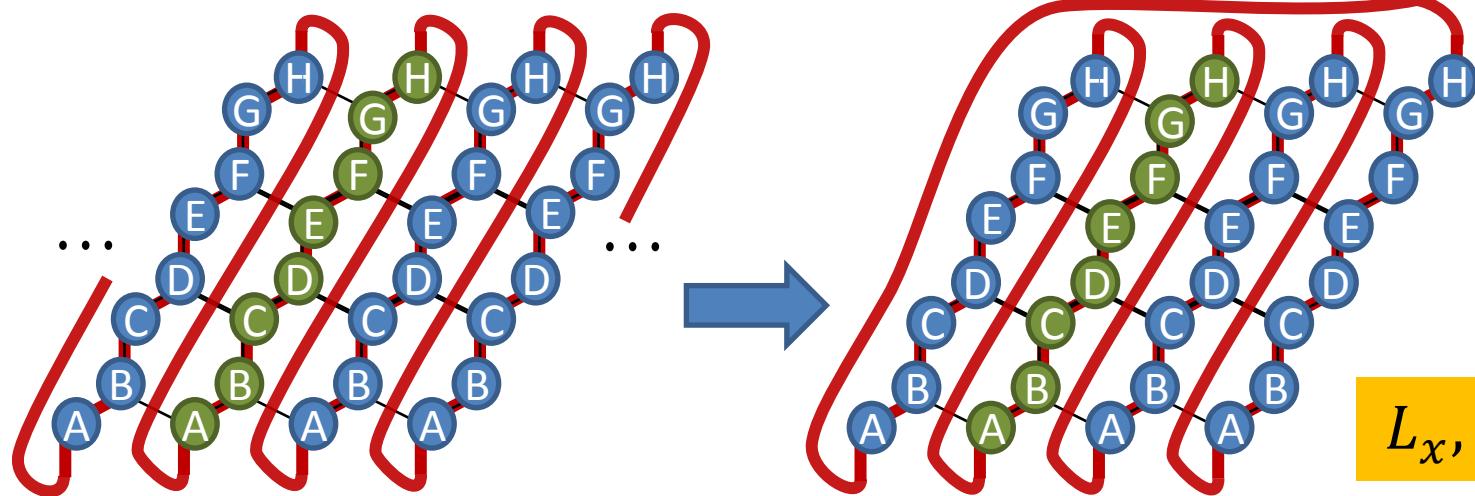
$\Psi_{\mathbb{S}}$



$\Psi_{\mathbb{I}}^{tor}$

$\Psi_{\mathbb{S}}^{tor}$

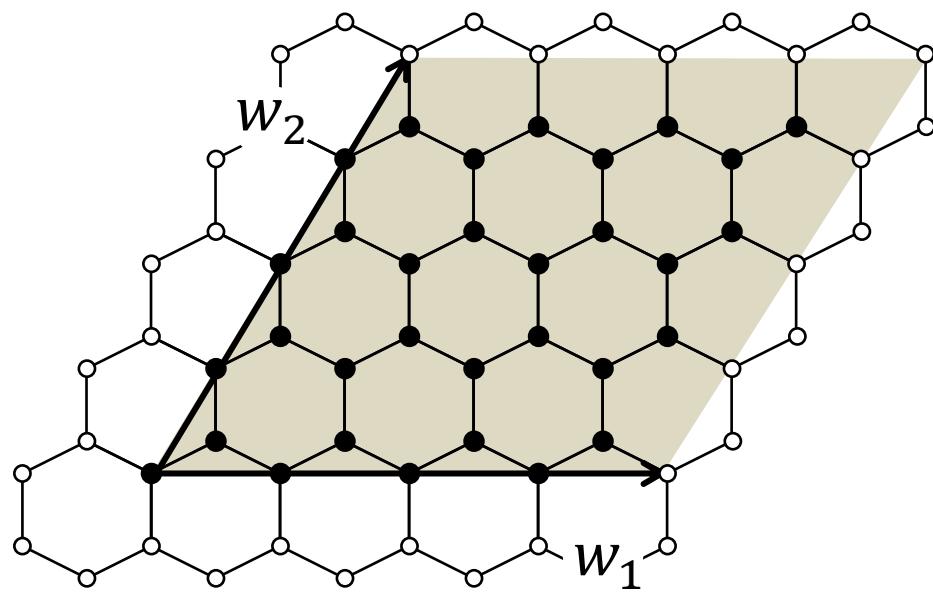
complete basis  
of quasi-degenerate  
ground subspace



$$L_x, L_y \gg \xi$$

$$(L_x = \infty, L_y = 4)$$

$$(L_x = 4, L_y = 4)$$



- torus: two vectors  $w_1, w_2$
- modular transformations  $SL(2, \mathbb{Z})$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$a, b, c, d \in \mathbb{Z}; ad - bc = 1$$

- generators

$$s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- ground space of  $H$  is a representation of the modular group

$$s \rightarrow S$$

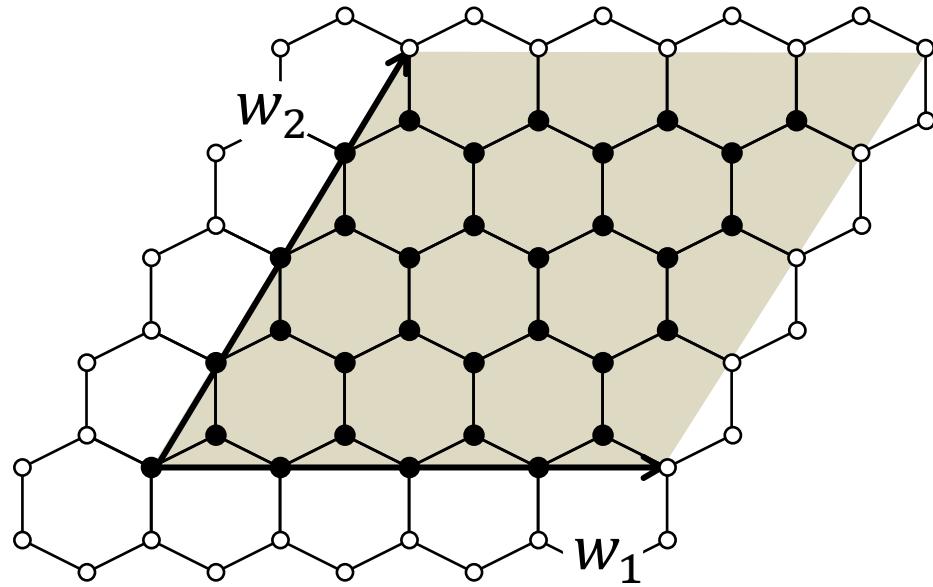
topological  $S$  matrix

$$S_{ij} = \frac{1}{D} \quad \begin{array}{c} \textcirclearrowleft \\ i \end{array} \quad \begin{array}{c} \textcirclearrowright \\ j \end{array}$$

$$u \rightarrow U$$

topological  $U$  matrix

$$U_{ii} = \frac{1}{d_i} \quad \begin{array}{c} \textcirclearrowleft \\ i \end{array} \quad \begin{array}{c} \textcirclearrowright \\ i \end{array}$$



- $\pi/3$  rotation  $R_{\pi/3}$  is a symmetry of  $H$  on torus
- it corresponds to  $US^{-1}$
- matrix of overlaps

$$V_{ij} = \langle \Psi_i^{tor} | R_{\pi/3} | \Psi_j^{tor} \rangle$$

$$V = DUS^{-1}D^\dagger$$

$$S = \begin{bmatrix} S_{\mathbb{II}\mathbb{II}} & S_{\mathbb{II}\mathbb{S}} \\ S_{\mathbb{S}\mathbb{II}} & S_{\mathbb{S}\mathbb{S}} \end{bmatrix} \quad U = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} 1 & 0 \\ 0 & \theta_{\mathbb{S}} \end{bmatrix} \quad D = \begin{bmatrix} e^{i\phi_{\mathbb{I}}} & 0 \\ 0 & e^{i\phi_{\mathbb{S}}} \end{bmatrix} \quad \begin{matrix} e^{i\phi_j} \text{ freedom} \\ \text{in defining } \Psi_j^{tor} \end{matrix}$$

$$S_{\mathbb{II}i}, S_{i\mathbb{II}} > 0$$

Numerics!

$$V = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} S_{\mathbb{II}\mathbb{II}} & S_{\mathbb{II}\mathbb{S}} e^{i(\phi_{\mathbb{S}} - \phi_{\mathbb{II}})} \\ S_{\mathbb{S}\mathbb{II}} e^{i(\phi_{\mathbb{II}} - \phi_{\mathbb{S}})} & \theta_{\mathbb{S}} (S_{\mathbb{S}\mathbb{S}})^* \end{bmatrix} = \begin{bmatrix} 0.685 + 0.181i & -0.229 + 0.669i \\ -0.693 - 0.138i & -0.183 + 0.681i \end{bmatrix}$$

$$L_x = L_y = 6$$

$$6 \times 6 \times 2 = 72 \text{ sites}$$

$$S = \begin{bmatrix} S_{\mathbb{II}} & S_{\mathbb{I}\mathbb{S}} \\ S_{\mathbb{S}\mathbb{I}} & S_{\mathbb{S}\mathbb{S}} \end{bmatrix} \quad U = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} 1 & 0 \\ 0 & \theta_{\mathbb{S}} \end{bmatrix} \quad D = \begin{bmatrix} e^{i\phi_{\mathbb{I}}} & 0 \\ 0 & e^{i\phi_{\mathbb{S}}} \end{bmatrix}$$

$$S_{\mathbb{I}i}, S_{i\mathbb{I}} > 0$$

Numerics!

$$V = e^{-i\frac{2\pi}{24}c} \begin{bmatrix} S_{\mathbb{II}} & S_{\mathbb{I}\mathbb{S}} e^{i(\phi_{\mathbb{S}} - \phi_{\mathbb{I}})} \\ S_{\mathbb{S}\mathbb{I}} e^{i(\phi_{\mathbb{I}} - \phi_{\mathbb{S}})} & \theta_{\mathbb{S}} (S_{\mathbb{S}\mathbb{S}})^* \end{bmatrix} = \begin{bmatrix} 0.685 + 0.181i & -0.229 + 0.669i \\ -0.693 - 0.138i & -0.183 + 0.681i \end{bmatrix}$$



topological  $S$  matrix

$$S_{ij} = \frac{1}{D} \quad i \circlearrowleft j \quad j$$

topological  $U$  matrix

$$U_{ii} = \frac{1}{d_i} \quad i \circlearrowleft \text{double loop} \quad i$$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} -1.4 & 0.2 \\ -1.4 & 4 + 4i \end{bmatrix}$$

$$U = e^{-i\frac{2\pi}{24}1} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times e^{-i\frac{2\pi}{24}0.01} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i0.007} \end{bmatrix}$$

chiral semion

(Monte Carlo statistical error  $< 10^{-4}$ )

Numerics!

topological  $S$  matrix

$$S_{ij} = \frac{1}{D} \quad i \begin{array}{c} \text{---} \\ \text{---} \end{array} \circlearrowleft \quad j$$

topological  $U$  matrix

$$U_{ii} = \frac{1}{d_i} \quad i \begin{array}{c} \text{---} \\ \text{---} \end{array} \circlearrowright \quad i$$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} -1.4 & 0.2 \\ -1.4 & 4 + 4i \end{bmatrix}$$

$$U = e^{-i\frac{2\pi}{24}1} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times e^{-i\frac{2\pi}{24}0.01} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i0.007} \end{bmatrix}$$

chiral semion

(Monte Carlo statistical error  $< 10^{-4}$ )

Numerics!

- from topological  $S$  matrix

- quantum dimensions  $d_{\mathbb{I}} = d_{\mathbb{S}} = 1, D = \sqrt{2}$
- $\mathbb{Z}_2$  fusion rules

$\mathbb{I} \times \mathbb{I} = \mathbb{I}$	$\mathbb{I} \times \mathbb{S} = \mathbb{S}$
$\mathbb{S} \times \mathbb{I} = \mathbb{S}$	$\mathbb{S} \times \mathbb{S} = \mathbb{I}$

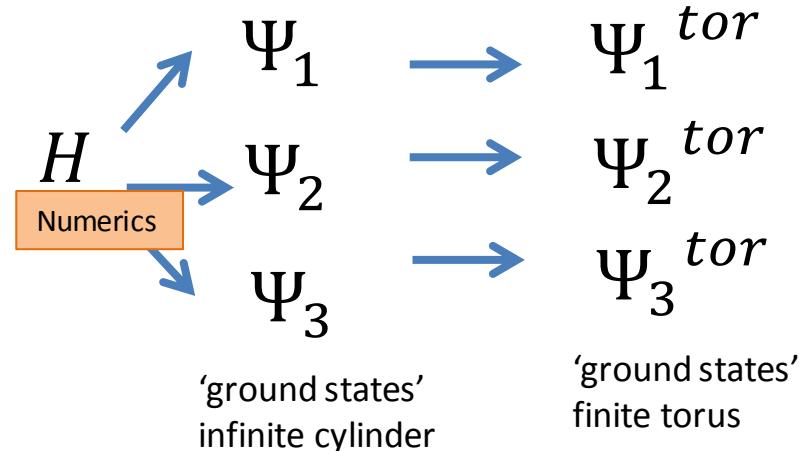
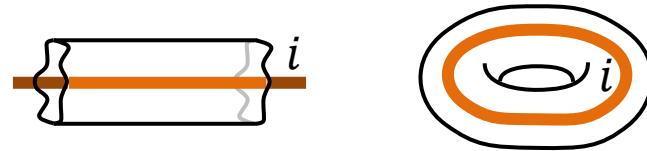
- from topological  $U$  matrix

- central charge  $c = 1$
- topological spin  $\Theta_{\mathbb{S}} = i$  (semion!)

Kitaev Honeycomb  
(non-Abelian phase with magnetic field)

A. Kitaev , Annals of Physics 2006

$$H = \sum_{\langle rr' \rangle_x} \sigma_r^x \sigma_{r'}^x + \sum_{\langle rr' \rangle_y} \sigma_r^y \sigma_{r'}^y + \sum_{\langle rr' \rangle_z} \sigma_r^z \sigma_{r'}^z + h \sum_r (\sigma_r^x + \sigma_r^y + \sigma_r^z)$$



Numerics!

$$S = \frac{1}{2} \begin{bmatrix} 1.02 & 1.40 & 1.01 \\ 1.41 & 0.03 & -1.41 \\ 1.04 & -1.36 & 1.04 \end{bmatrix}$$

$$L_x = L_y = 4$$

$$\approx \frac{1}{2} \begin{bmatrix} 1.00 & 1.41 & 1.00 \\ 1.41 & 0.00 & -1.41 \\ 1.00 & -1.41 & 1.00 \end{bmatrix} \quad \sqrt{2} \approx 1.41$$

5% !

Ising anyon model

$$S = \frac{1}{2} \begin{bmatrix} \mathbb{I} & \sigma & \varepsilon \\ 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

- quantum dimensions

$$d_{\mathbb{I}} = 1 \quad d_{\sigma} = \sqrt{2} \quad d_{\varepsilon} = 1; \quad D = 2$$

- fusion rules

$$\sigma \times \varepsilon = \sigma \quad \sigma \times \sigma = \mathbb{I} + \varepsilon \quad \varepsilon \times \varepsilon = \mathbb{I}$$

# OUTLINE

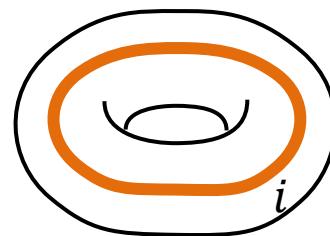
## 1) GROUND STATES

Infinite cylinder



- edge spectrum
  - quantum dimensions
  - chiral CFT

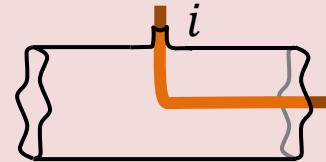
Finite torus



- S matrix
- U matrix
- mutual statistics
- central charge
- quantum dimensions
- topological spins
- fusion rules

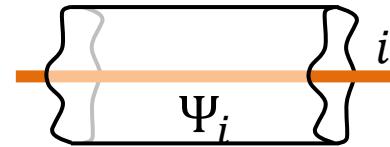
## 2) QUASIPARTICLE EXCITATIONS

Infinite cylinder

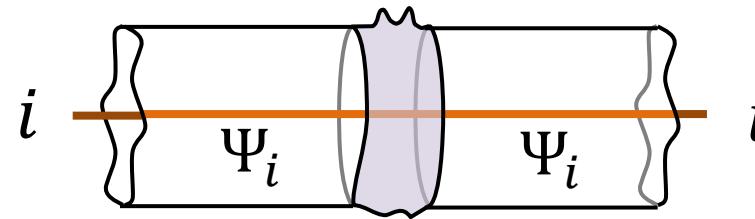


- integer excitations
- fractionalized excitations

- ground states:

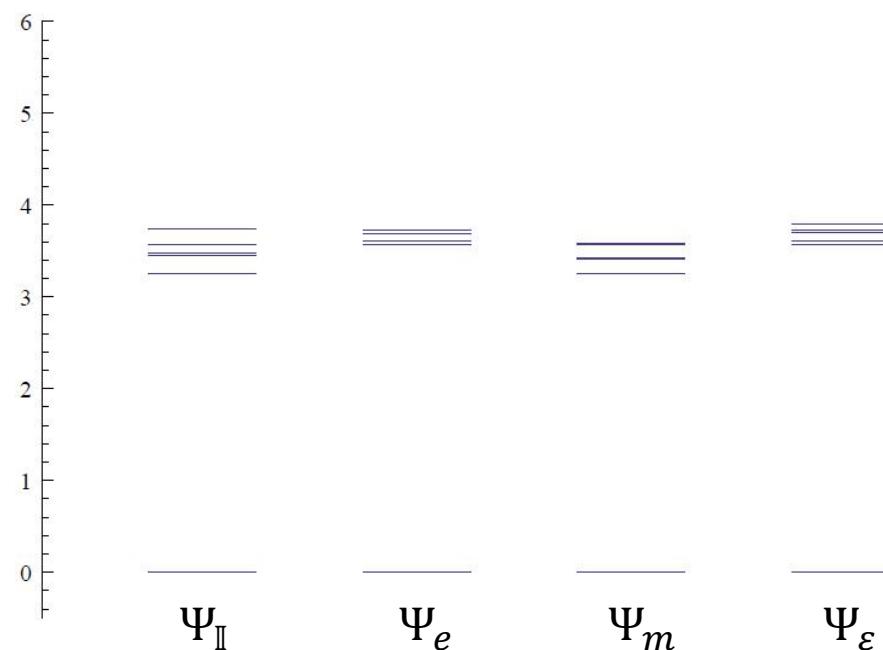


- integer excitations

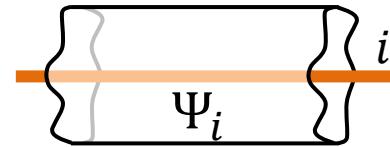


Example: toric code with magnetic field  $0.1\sigma_z + 0.05\sigma_x$

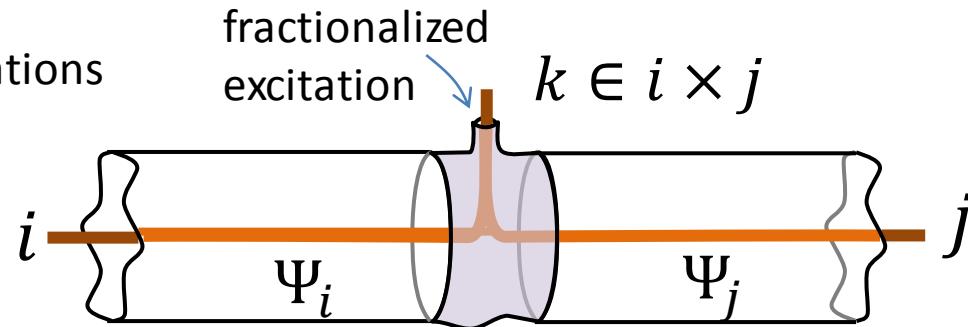
$$i = \mathbb{I}, e, m, \varepsilon$$



- ground states:

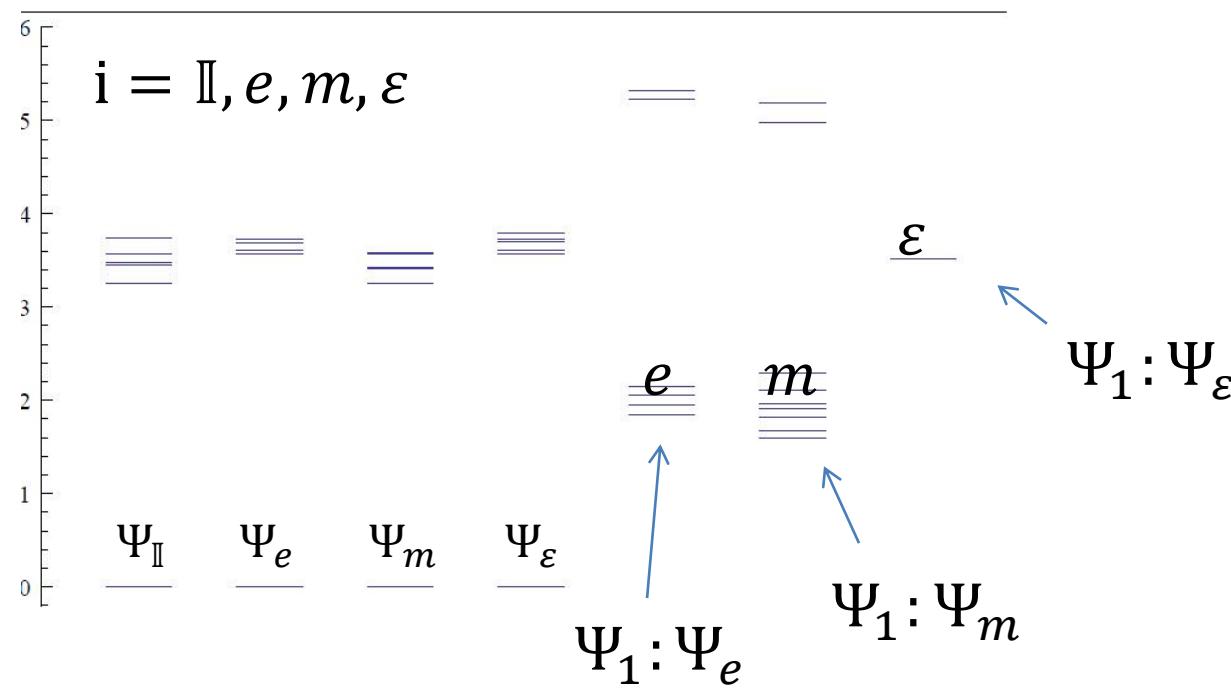


- fractionalized excitations

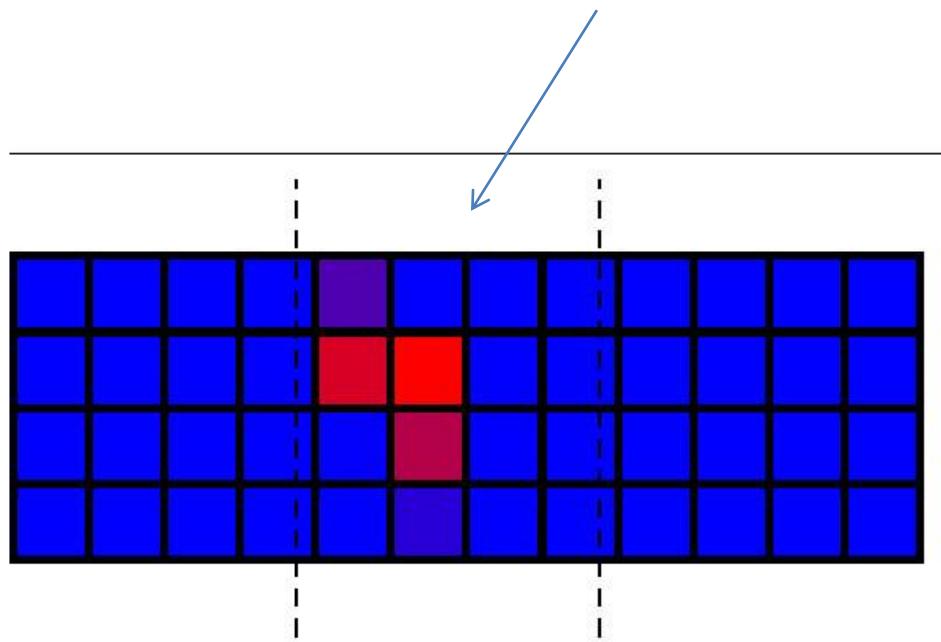


Example: toric code with magnetic field

$0.1\sigma_z + 0.05\sigma_x$

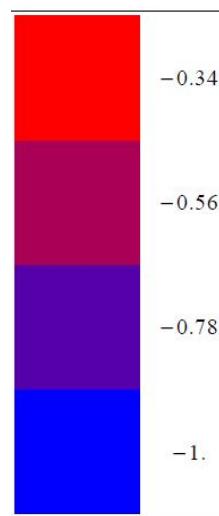


e



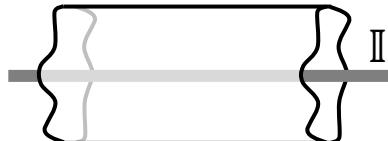
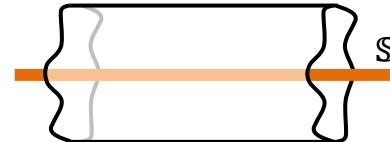
$\Psi_1$

$\Psi_e$

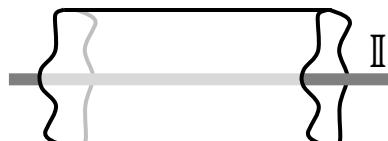
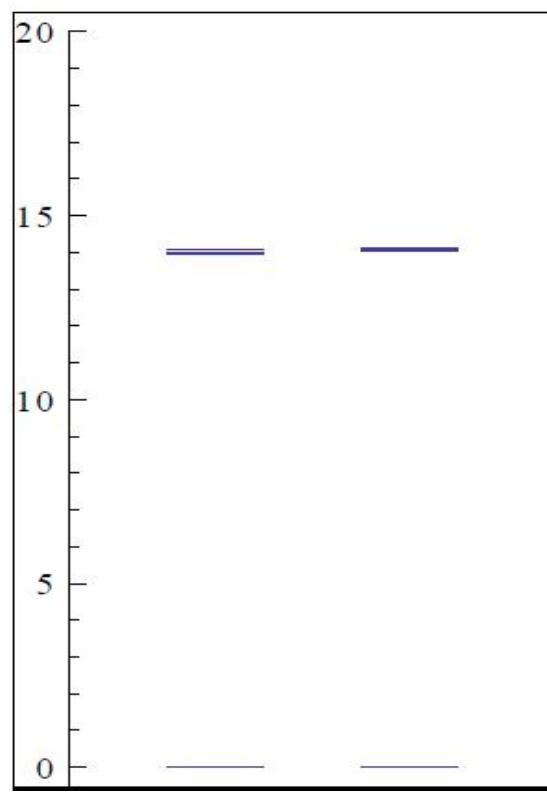
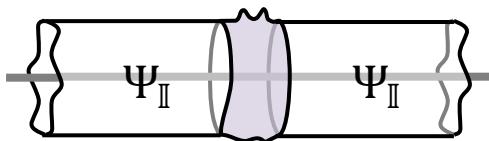
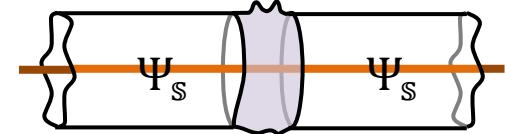


# Haldane model (hard-core bosons)

- ground states:

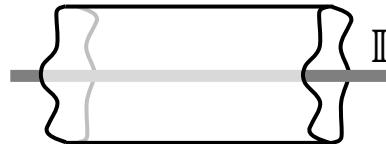
 $\Psi_{\mathbb{I}}$  $\Psi_{\mathbb{S}}$ 

- integer excitations

 $\Psi_{\mathbb{I}}$  $\Psi_{\mathbb{S}}$

# Haldane model (hard-core bosons)

- ground states:

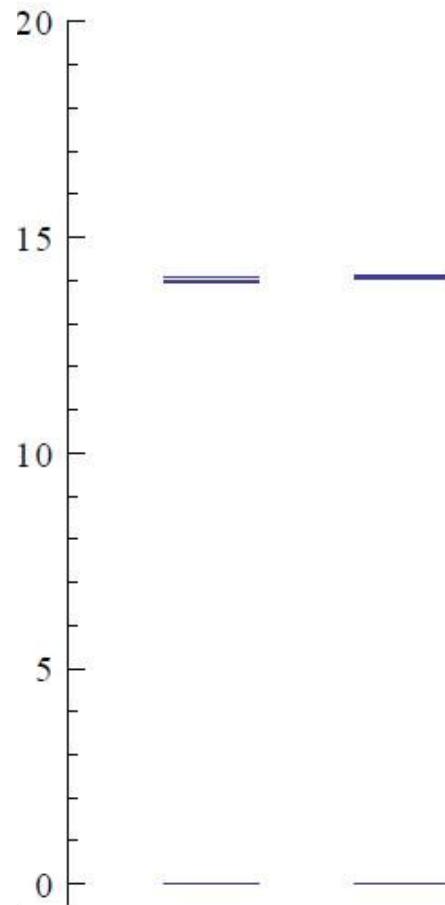


$$\Psi_{\amalg}$$

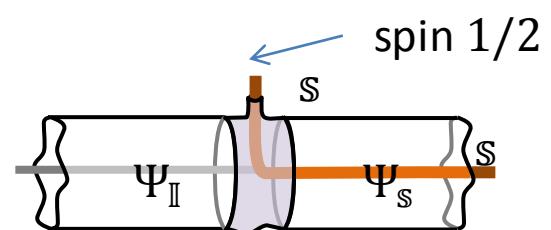


$$\Psi_{\$}$$

- fractionalized excitations



≡



## microscopic Hamiltonian

$H$

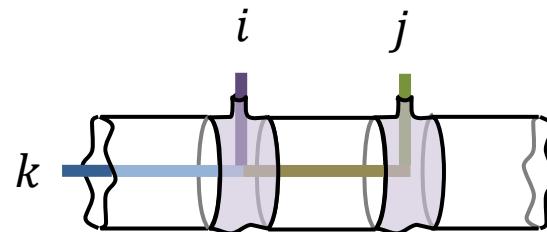
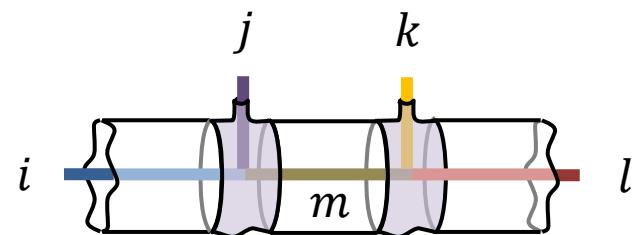
on infinite cylinder

$$i \quad j \quad k \\ m \quad \quad \quad l \\ \quad \quad \quad \quad = \sum_n [F^{ijk}]_l {}_{mn}$$

- F – symbols

$$i \quad j \\ \quad \quad \quad k \\ \quad \quad \quad \quad = R^{ij} {}_k$$

- R – symbols



ultimate goal:  
complete characterization

## SUMMARY

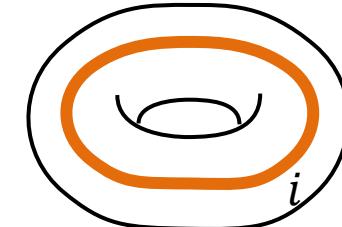
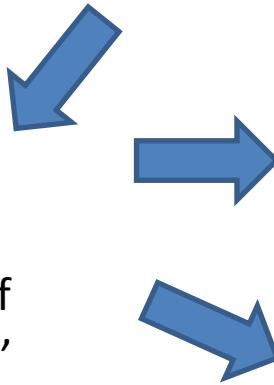
microscopic Hamiltonian

$$H$$

on infinite cylinder

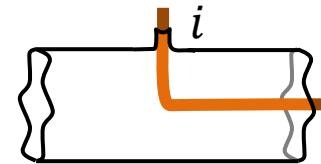


$\{\Psi_i\}$  complete set of  
'ground states'



$\{\Psi_i^{tor}\}$  complete basis in  
quasi-degenerate  
ground space

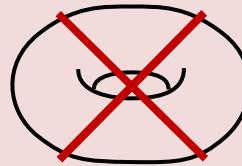
- integer excitations
- fractionalized excitations



Thank you !

## FAQs:

Q: Why on the cylinder (and not on the torus)?

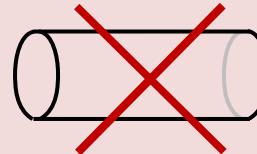


A1: Cost of DMRG/tensor networks is much lower

A2: Simpler entanglement spectrum

A3: Single fractionalized excitation

Q: Why on the infinite cylinder (and not on a large cylinder)?



A1: complete set of ‘ground states’

A2: translation invariance/unit cell: map to torus

Q: Why DMRG (and not 2D tensor network, e.g. PEPS)?

A: DMRG is better understood and more reliable;  
but  $L_y$  limited. PEPS is next.