Holographic Geometry in Entanglement Renormalization

Shinsei Ryu
Univ. of Illinois, Urbana-Champaign

Collaborators Masahiro Nozaki (Kyoto)
Tadashi Takayanagi (Kyoto)

Thanks: Yi Chen Hu

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Entanglement Entropy in QFTs

(i) bipartition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

(ii) take partial trace $\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|$

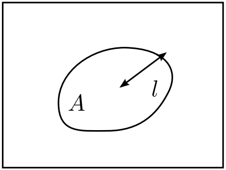
$$\rho_A = \mathrm{tr}_B|\Psi\rangle\langle\Psi| = \sum_j p_j|\psi_j\rangle_A\langle\psi_j|_A \quad \left(\sum_j p_j = 1\right)$$

(iii) entanglement entropy

$$\boxed{S_A = -\mathrm{tr}_A\left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j}$$

application to many-body systms and field theories:

A, B : submanifold of the total system



motivation for entanglement entropy

EE can be a good "order parameter" for quantum systems (?)

- defined purely in terms of wavefunctions

  (i.e., always possible to define;
  EE measures a response to external gravity)

$$S_A = \frac{c}{3}\log(l/a) + O(1)$$

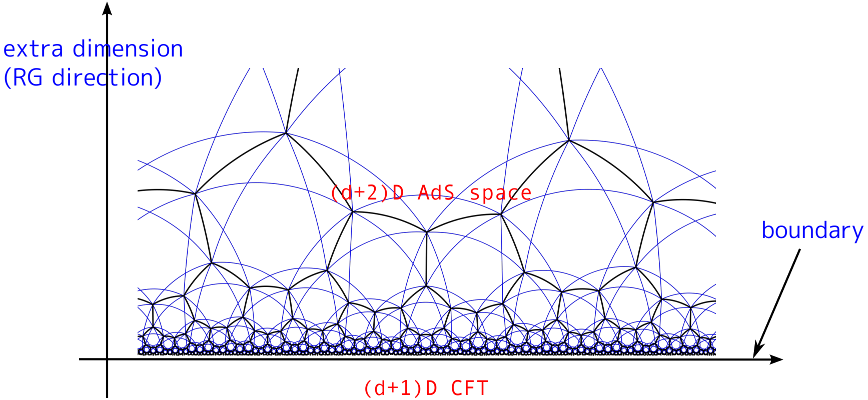
- use computational complexity to classify quantum states ?

- EE spectrum: new tool to classify symmetry protected gapped phases

not sure how to measure it

rather difficult to compute !

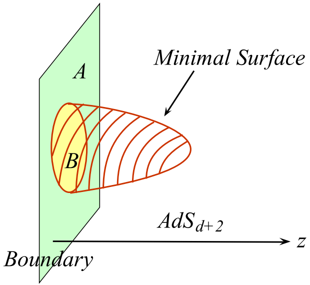
AdS/CFT


extra dimension
(RG direction)
(d+2)D AdS space
boundary
(d+1)D CFT

Geometry <--> Entanglement

- Holographic formula for EE [SR-Takayanagi (06)]

$$\boxed{S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}}$$



Applications & Checks:

- area law
- EE of an interval in 1+1 d CFT
- a log part of EE in even dim CFT
- agrees with BH formula for high T
- (in)equalities, strong subadditivity,
- two intervals (mutual information)
- proof when A = ball (dA = sphere)

Introduction

- Holography --> "geometrization" of RG

- Entanglement and holography

$$S_A = \frac{\mathrm{Area}}{4G_N}$$

Geometry --> Entanglement

- Due to "duality"

Entanglement --> Geometry

- MERA: Entanglement and real space RG

tensor network approach to quantum manybody systems

- Representing many-body wavefunctions by contracting many tensors
  DMRG, MPS, MERA, PEPS, etc.

$$|\Psi\rangle = \sum_{s_1,s_2,s_3,s_4\cdots} C^{s_1,s_2,s_3,s_4\cdots}|s_1,s_2,s_3,s_4\ldots\rangle$$

product state:

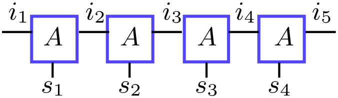
$$|\Psi\rangle = \sum_{\{s_a\}} A^{s_1}A^{s_2}A^{s_3}\cdots|s_1,s_2,s_3,s_4\ldots\rangle = \prod_i \sum_{s_i} A^{s_i}|s_i\rangle$$

EE = 0

physical degrees of freedom

MPS (matrix product state) :

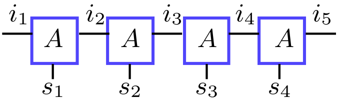
$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2}A^{s_2}_{i_2,i_3}A^{s_3}_{i_3,i_4}A^{s_4}_{i_4,i_5}\cdots|s_1,s_2,s_3,s_4\ldots\rangle$$



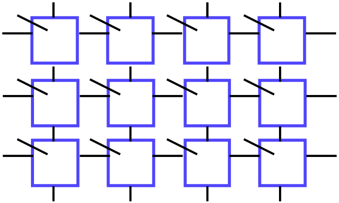
auxiliary index

MPS (matrix product state) :

$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2} A^{s_2}_{i_2,i_3} A^{s_3}_{i_3,i_4} A^{s_4}_{i_4,i_5} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle$$

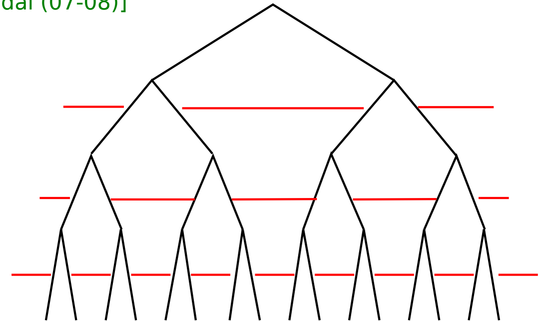


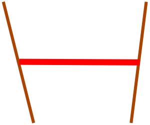
PEPS (projected entangled pair state) :



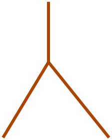
multiscale entanglement renormalization ansatz (MERA)

[Vidal (07-08)]

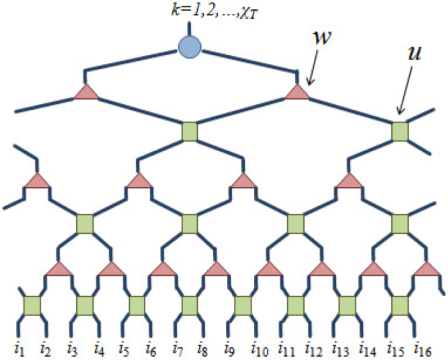




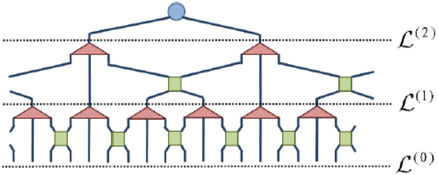
"disentangler"



isometry
(coarse-graining)



k=1,2,...,χT
w
u
i1 i2 i3 i4 i5 i6 i7 i8 i9 i10 i11 i12 i13 i14 i15 i16



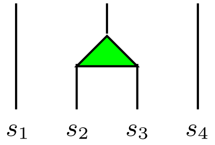
$\mathcal{L}^{(2)}$
$\mathcal{L}^{(1)}$
$\mathcal{L}^{(0)}$

block spin decimation and disentangler

- Block spin decimation

$$\rho_{tot} = |\Psi\rangle\langle\Psi|$$

$$\rho_{23} = \mathrm{Tr}_{14}\rho_{tot} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$$



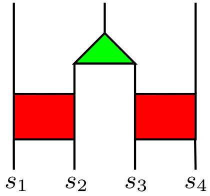
small p_i --> throw away

- Disentangler

$$|\Psi\rangle = \frac{(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)_{12}}{\sqrt{2}} \frac{(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)_{34}}{\sqrt{2}} \qquad p_i = \frac{1}{4}(\forall i)$$

$$U\frac{(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)_{12}}{\sqrt{2}} = |\uparrow\uparrow\rangle_{12}$$

$$\mathrm{Tr}_{14}\left[U_{12}\otimes U_{34}\rho_{tot}(U_{12}\otimes U_{34})^\dagger\right]$$



tensor network approach to quantum manybody systems

- Targeting quantum manybody problems according to the scaling of the entanglement entropy

- Classification of quantum phases in terms of computational complexity

(partial and biased list)

gapped phases

- QHE
- (topological) insulators

MPS, PEPS class

quantum critical

- QCP
- Tomonaga-Luttinger liquid

MERA class

metallic phases (itinerant system)

- Fermi liquid
- non-Fermi liquid (?)

branching MERA

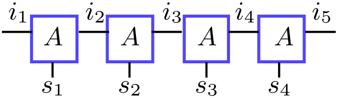
classical phases

symmetry breaking
finite T transition
no entanglement

structure of tensor-network and entanglement entropy

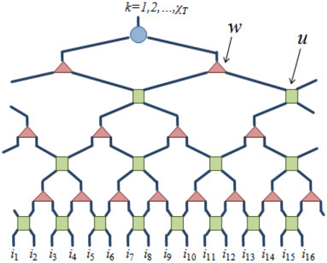
matrix product state (DMRG):

$$|\Psi\rangle = \sum_{\{s_a\}} \sum_{\{i_n=1,\cdots,\chi\}} A^{s_1}_{i_1,i_2} A^{s_2}_{i_2,i_3} A^{s_3}_{i_3,i_4} A^{s_4}_{i_4,i_5} \cdots |s_1, s_2, s_3, s_4 \ldots\rangle$$



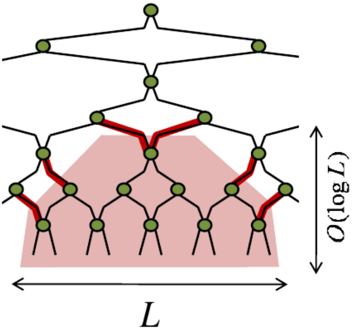
EE: $S_A \leq \log \chi$

MERA



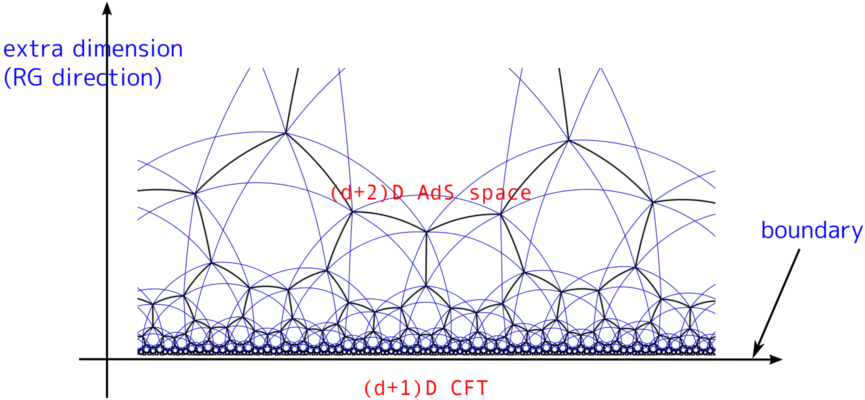
MERA and holographich entanglement entropy

[Swingle (09)]



EE: $S_A \sim \log(l/a)$

MERA and holography ?

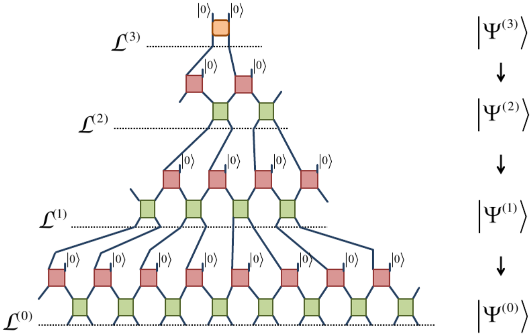


Can we make "AdS/MERA" more precise ?

MERA and quantum circuit

- Tensor network method can be formulated as a quantum circuit (successive applications of unitary transformations)

- For MERA: add dummy states |0>



continuous MERA (cMERA)

[Haegeman-Osborne-Verschelde-Verstraete (11)]

- Quantum circuit representation of the target ground state:

$$|\Psi(u_{\rm IR})\rangle \equiv |\Omega\rangle \qquad\qquad |\Psi(u_{\rm UV})\rangle \equiv |\Psi\rangle$$

$$|\Psi(u)\rangle = U(u, u_{\rm IR})|\Omega\rangle$$

$$|\Psi\rangle = U(0, u)|\Psi(u)\rangle$$

$$u_{\rm UV} = 0$$

$$u_{\rm IR} = -\infty$$

- MERA evolution operator

$$U(u_1, u_2) = P\exp\left[-i\int_{u_2}^{u_1}(\underbrace{K(u)}_{\text{disentangler}} + \underbrace{L}_{\text{coarse-graining}})du\right]$$

- Optimizing |Omega>, U --> true ground state

cMERA for free boson

- free boson in d+1 dim:

$$H = \frac{1}{2}\int d^dk\,[\pi(k)\pi(-k) + \epsilon_k^2 \cdot \phi(k)\phi(-k)]$$

$$\phi(k) = \frac{a_k + a^\dagger_{-k}}{\sqrt{2\epsilon_k}} \qquad \pi(k) = \sqrt{2\epsilon_k}\left(\frac{a_k - a^\dagger_{-k}}{2i}\right)$$

- IR state:

$$\left(\sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x)\right)|\Omega\rangle = 0$$

completely uncorrelated

$$(\alpha_k a_k + \beta_k a^\dagger_{-k})|\Omega\rangle = 0$$

$$\alpha_k = \frac{1}{2}\left(\sqrt{\frac{M}{\epsilon_k}} + \sqrt{\frac{\epsilon_k}{M}}\right) \qquad \beta_k = \frac{1}{2}\left(\sqrt{\frac{M}{\epsilon_k}} - \sqrt{\frac{\epsilon_k}{M}}\right)$$

- coarse-graining

$$e^{-iuL}\phi(k)e^{iuL} = e^{-\frac{d}{2}u}\phi(e^{-u}k)$$

$$e^{-iuL}\pi(k)e^{iuL} = e^{-\frac{d}{2}u}\pi(e^{-u}k)$$

- disentangler

$$K(u) = \frac{1}{2}\int d^dk\,[g(k,u)(\phi(k)\pi(-k)+\pi(k)\phi(-k))]$$

$$g(k,u) = \chi(u)\cdot\Gamma\left(|k|/\Lambda\right)$$

cutoff function

- variational principle:

$$E = \langle\Psi|H|\Psi\rangle = \langle\Omega|H(u_{\mathrm{IR}})|\Omega\rangle$$

$$\chi(u) = \frac{1}{2}\cdot\frac{e^{2u}}{e^{2u}+m^2/\Lambda^2}, \quad M = \sqrt{\Lambda^2+m^2}$$

- scale-dependent Bogoliubov transformation:

$$(\alpha_k, \beta_k) \cdot \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} |\Omega\rangle = 0 \qquad \text{IR}$$

$$U(u)(\alpha_k, \beta_k) \cdot \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} |\Omega\rangle = 0$$

$$(\alpha_k, \beta_k) \cdot U(u) \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} U^\dagger(u) \cdot U(u)|\Omega\rangle = 0$$

$$(\alpha_k, \beta_k) \cdot M(u) \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} |\Psi(u)\rangle_L = 0$$

$$(A_k, B_k) \cdot \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} |\Psi(u)\rangle_L = 0 \qquad \text{UV}$$

Bures distance (quantum distance)

- Bures distance:

$$D_{\mathrm{B}}(\rho_1, \rho_2) := 2\left(1 - \mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right)$$

- for pure states: $\rho_1 = |\psi_1\rangle\langle\psi_1| \qquad \rho_2 = |\psi_2\rangle\langle\psi_2|$

$$D_{\mathrm{B}}(\psi_1, \psi_2) = 2\left(1 - |\langle\psi_1|\psi_2\rangle|\right)$$

- for infinitesimally close state:

$$D_{\mathrm{B}}[\psi(\xi), \psi(\xi + d\xi)] = g_{ij}(\xi) d\xi_i d\xi_j$$

$$\boxed{g_{ij}(\xi) = \mathrm{Re}\,\langle\partial_i\psi(\xi)|\partial_j\psi(\xi)\rangle - \langle\partial_i\psi(\xi)|\psi\rangle\langle\psi|\partial_j\psi(\xi)\rangle}$$

Introducing metric in MERA

- Proposal for a metric in radial direction:

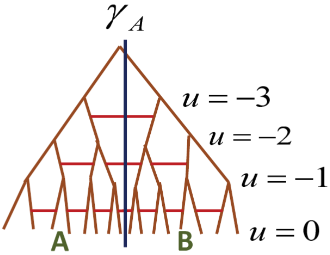
$$\boxed{g_{uu}(u)du^2 = \mathcal{N}^{-1}\left(1 - |{}_L\langle\Psi(u)|\Psi(u+du)\rangle_L|^2\right)}$$

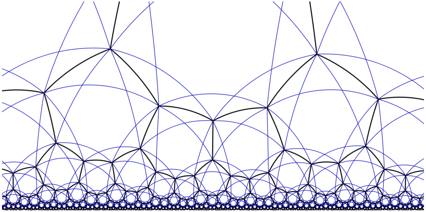
where

$$|\Psi(u+du)\rangle_L = e^{iLu}|\Psi(u)\rangle$$ wfn in "interaction picture"

$$\mathcal{N} = \mathrm{Vol.}\int_{|k|\leq\Lambda e^u} d^dk$$ normalization

Motivation for the metric





$$S_A \propto L^{d-1} \sum_{u=-\infty}^{0} n(u) \cdot 2^{(d-1)u}$$

strength of disentangler

$$S_A = \frac{1}{4G_N} \cdot \frac{V_{d-1}}{\epsilon^{d-1}} \int_{u_{\rm IR}(=-\infty)}^{u_{\rm UV}(=0)} du \sqrt{g_{uu}} e^{(d-1)u}$$

$$ds^2 = g_{uu} du^2 + \frac{e^{2u}}{\epsilon^2} d\vec{x}^2 + g_{tt} dt^2$$

- relativistic free scalar:

$$ds^2 = g_{uu} du^2 + \frac{e^{2u}}{\epsilon^2} d\vec{x}^2 + g_{tt} dt^2$$

$$g_{uu}(u) = \chi(u)^2 = \frac{e^{4u}}{4(e^{2u} + m^2/\Lambda^2)^2}$$

massless limit:

$$g_{uu}(u) = \text{const.}$$

AdS metric

massive case:

$$e^{2u} = \frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}$$

AdS soliton

$$ds^2 = \frac{dz^2}{4z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right) dx^2 + g_{tt} dt^2$$

- flat space:

$$H = \int d^d x\, \phi(x) e^{A(-\partial^2)^{w/2}} \phi(x)$$

$$\epsilon_k \propto e^{A \cdot k^w}$$

$$g_{uu}(u) = g(u)^2 \propto e^{2wu}$$

c.f. Li-Takayanagi (10)

Issues

- Where is the Einstein equation ? [Swingle (12)]
- Large-N ? higher spin ? [Douglas, Mazzucato, Razamat (10)]
- Diffeo invariance ?
  c.f. MERA for t-dependent process [Matsueda (12)]

- Finite temperature ?
- Time-component of metric g_tt ?
- Effects of interactions ?

Advantages of AdS/MERA:

- No need for large-N
- Can define geometry for generic many-body states

Time-dependence

- Time-dependent excited state: $(A_k a_k + B_k a^\dagger_{-k})|\Psi_{ex}\rangle = 0$

$$A_k = \sqrt{1 + a_k^2} \cdot e^{i\epsilon_k t + i\theta_k},$$

$$B_k = a_k \cdot e^{-i\epsilon_k t + i\theta_k},$$

- Phase ambiguirty -- Psi is indep. of theta, but metric is not

- Choose theta s. t. the extremal surface is on the corresponding time-slice.

Topological phases

- Closing comment: topological phases

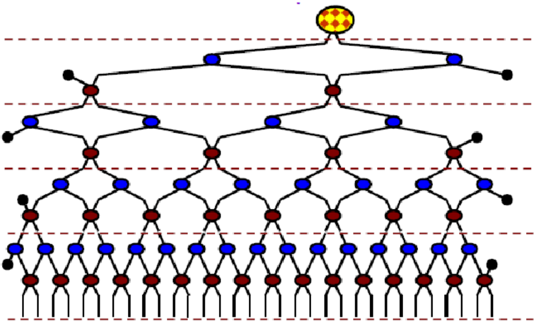
  e.g. phases in condensed matter that are described by
  the Chern-Simons theory

- no classical order parameter,
  highly entangled quantum states of matter

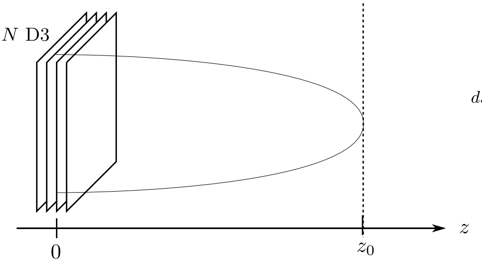
MERA for topological phase

Topological infomation is strored in "top tensor"

[Koenig, Reichardt, Vidal (08)]
[Aguado-Vidal (09)]



AdS/CFT, AdS/CS

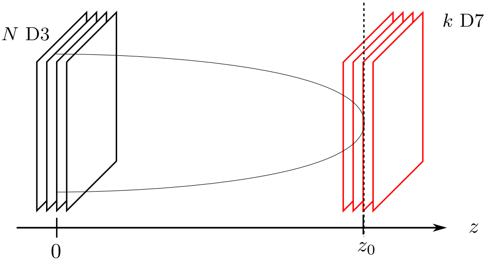


holographic dual of pure YM in (2+1)D

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + f(z)dy^2 + dx_1^2 + dx_2^2 + f^{-1}(z)dz^2\right)$$

$$f(z) = 1 - (z/z_0)^4$$

[Witten (98)]



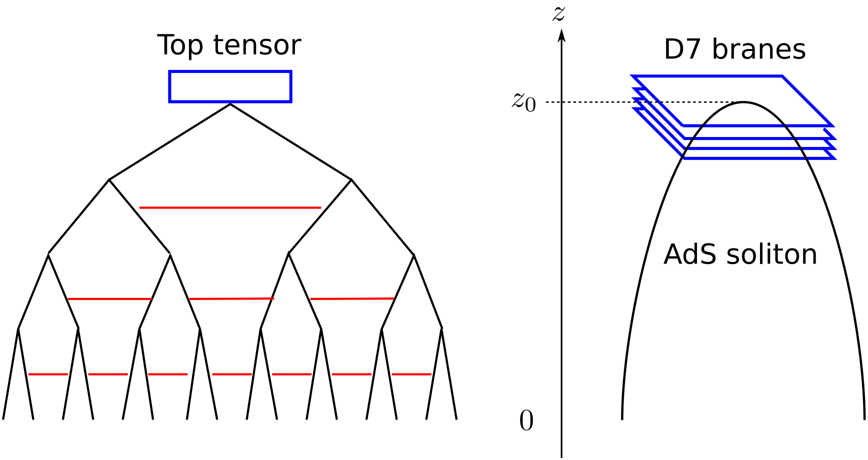
holographic dual of FQHE

$$S_{\mathrm{D3}} = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

$$S_{top} \sim \frac{k^2}{2}\log N$$

[Fujita-Li-Ryu-Takayanagi (10)]

D7-brane = "Top tensor" ?



summary and outlook

- Metric for tensor MERA representation of quantum states

- Behaviors expected from AdS/CFT

- Classical phases of matter <-- group theory (symmetry breaking)
  Quantum phases of matter <-- geometry (entanglement)
  Topological phases <--> D-branes

- Use of entanglement entropy/spectrum has been increasingly important in solving/characterizing many-body quantum systems