From Renormalization Group to Emergent Gravity : holographic description of quantum many-body systems

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Different types of gapless phases

- IR free theory : gauge theory in D>4
- Weakly interacting theory : 3D O(N) model in the large N limit
- Theory that has no weak coupling description : 4D gauge theory at an intermediate coupling
- Strongly coupled in original variables but weakly coupled in dual variables : strongly coupled 4D gauge theory (electro-magnetic duality)
  - Fractionalization in large N limit : new quantum number
  - Holography : new dimension

Holography : AdS/CFT correspondence

[Maldacena]

- D-dim QFT is dual to (D+1)-dim gravitational theory
  - N=4 SU(N) gauge theory in 4D = IIB superstring theory in $AdS_5 x S^5$
  - Weak coupling description for strongly coupled QFT for a large N
- The correspondence has been used to reproduce many phenomena in condensed matter systems : hydrodynamics, superconductivity, non-Fermi liquid (mostly phenomenological with a few exceptions)

Dictionary in the duality

[Gubser, Klebanov, Polyakov; Witten]

Low energy

(D+1)-dim space

$$O_n(x) \leftrightarrow j_n(x, z)$$

$$T_{\mu\nu}(x) \leftrightarrow g_{\mu\nu}(x, z)$$

z

High energy

x

D –dim flat space

$$\int D\phi(x) e^{iS_D[\phi(x)] + i\int J_n(x) O_n} = \int Dj(x,z) e^{iS_{D+1}[j(x,z)]}\Big|_{j_n(x,z=0)=J_n(x)}$$

$$\rightarrow e^{iS_{D+1}[\bar{j}(x,z)]}\Big|_{\bar{j}_n(x,z=0)=J_n(x)}$$

The bulk theory includes gravity

Low energy

(D+1)-dim space

$$O_n(x) \leftrightarrow j_n(x,z)$$

z

$$T_{\mu\nu}(x) \leftrightarrow g_{\mu\nu}(x,z)$$

High energy

x

D –dim flat space

$$\begin{aligned} S_{D+1} &= \int d^{D+1}x\sqrt{G}[C_0 + R^{D+1} + \ldots] \\ &= \int dz \int d^D x \left[\pi_{\mu\nu}\partial_z g^{\mu\nu} + \sqrt{g}(C_0 + R^D + \frac{g^{-1}}{2}\pi^{\mu\nu}\pi_{\mu\nu} + \ldots)\right] \end{aligned}$$

- Holography is believed to be a general framework for a large class of QFT's

[Das, Jevicki; Gopakumar; Heemskerk, Penedones, Polchinski; Lee; Faulkner, Liu, Rangamani; Douglas, Mazzucato, Razamat,…]

- No first-principle derivation for the conjecture : no systematic way to derive the dual theory for a given QFT / quantum many-body system
- This talk : An explicit construction of the gravitational dual from QFT/quantum many-body system

Step 0 : Partition function is functional of spacetime dependent sources

$$Z[J(x)] = \int D\phi \; e^{i\int dx \mathcal{L}}$$

$$\mathcal{L} = -J_n(x)O_n + J_{mn}(x)O_mO_n + \ldots$$

- O_n : set of `fundamental' operators
- Any local operator allowed by symmetry can be written as a polynomial of fundamental operators and their derivatives

e.g.

$$T_{\mu\nu} \sim tr(\phi\partial_\mu\partial_\nu\phi)$$

$$\mathrm{tr}(\phi\partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_i}\phi\partial_{\nu_1}\partial_{\nu_2}...\partial_{\nu_j}\phi...) \quad \text{for matrix field}$$

Step 1 : remove non-fundamental operators by introducing auxiliary fields

$$Z[J(x)] = \int Dj_n^{(1)} Dp_n^{(1)} D\phi \; e^{i\int dx \mathcal{L}'}$$

$$\mathcal{L}' = j_n^{(1)}(p_n^{(1)} - O_n) - J_n p_n^{(1)} + J_{nm} p_n^{(1)} p_m^{(1)} + \ldots$$

- $J_n^{(1)}$: Lagrangian multiplier that plays the role of dynamical source that enforces the constraint $p_n^{(1)} = O_n$
- $P_n^{(1)}$: dynamical operator

$$g^{(1)\mu\nu} tr(\phi \partial_\mu \partial_\nu \phi) \qquad J^{\mu\nu\lambda\sigma} \pi^{(1)}_{\mu\nu} \pi^{(1)}_{\lambda\sigma}$$

Step 2 : Integrate out high energy mode

$$\phi_< : |k| < \Lambda e^{-dz}, \quad \phi_> : \Lambda e^{-dz} < |k| < \Lambda$$

$$Z[J(x)] = \int Dj_n^{(1)} Dp_n^{(1)} D\phi_< \, e^{i\int dx \mathcal{L}''}$$

$$\mathcal{L}'' = J_{nm} p_n^{(1)} p_m^{(1)} + p_n^{(1)} (j_n^{(1)} - J_n) + dz \mathcal{L}_c[j^{(1)}]$$

$$-(j_n^{(1)} + dz A_n[j^{(1)}]) O_n + dz B_{nm}[j^{(1)}] O_n O_m$$

- Casimir energy $\mathcal{L}_c[j^{(1)}] \sim C_0 + R^D[g^{(1)\mu\nu}] + \ldots$
- Quantum correction for fundamental operators
- Quadratic term of fundamental operators

Step 3 : remove non-fundamental operators by introducing a second set of auxiliary fields

$$Z[J(x)] = \int Dj_n^{(1)} Dp_n^{(1)} Dj_n^{(2)} Dp_n^{(2)} D\phi_< \, e^{i\int dx \mathcal{L}'''}$$

$$\mathcal{L}''' = J_{nm} p_n^{(1)} p_m^{(1)} + p_n^{(1)} (j_n^{(1)} - J_n) + dz \mathcal{L}_c[j_n^{(1)}]$$

$$+ j_n^{(2)} (p_n^{(2)} - O_n) - (j_n^{(1)} + dz A_n[j^{(1)}]) p_n^{(2)} + dz B_{nm}[j^{(1)}] p_n^{(2)} p_m^{(2)}$$

- Low energy fields has only fundamental operators
- Quadratic term in p_n $\quad B^{\mu\nu\lambda\sigma} \pi^{(2)}_{\mu\nu} \pi^{(2)}_{\lambda\sigma}$

Step 4 : repeat 2-3 again and again

$$Z[J(x)] = \int \prod_{l=1}^{\infty} Dj_n^{(l)}(x) Dp_n^{(l)}(x)\ e^{i\int d^D x \mathcal{L}}$$

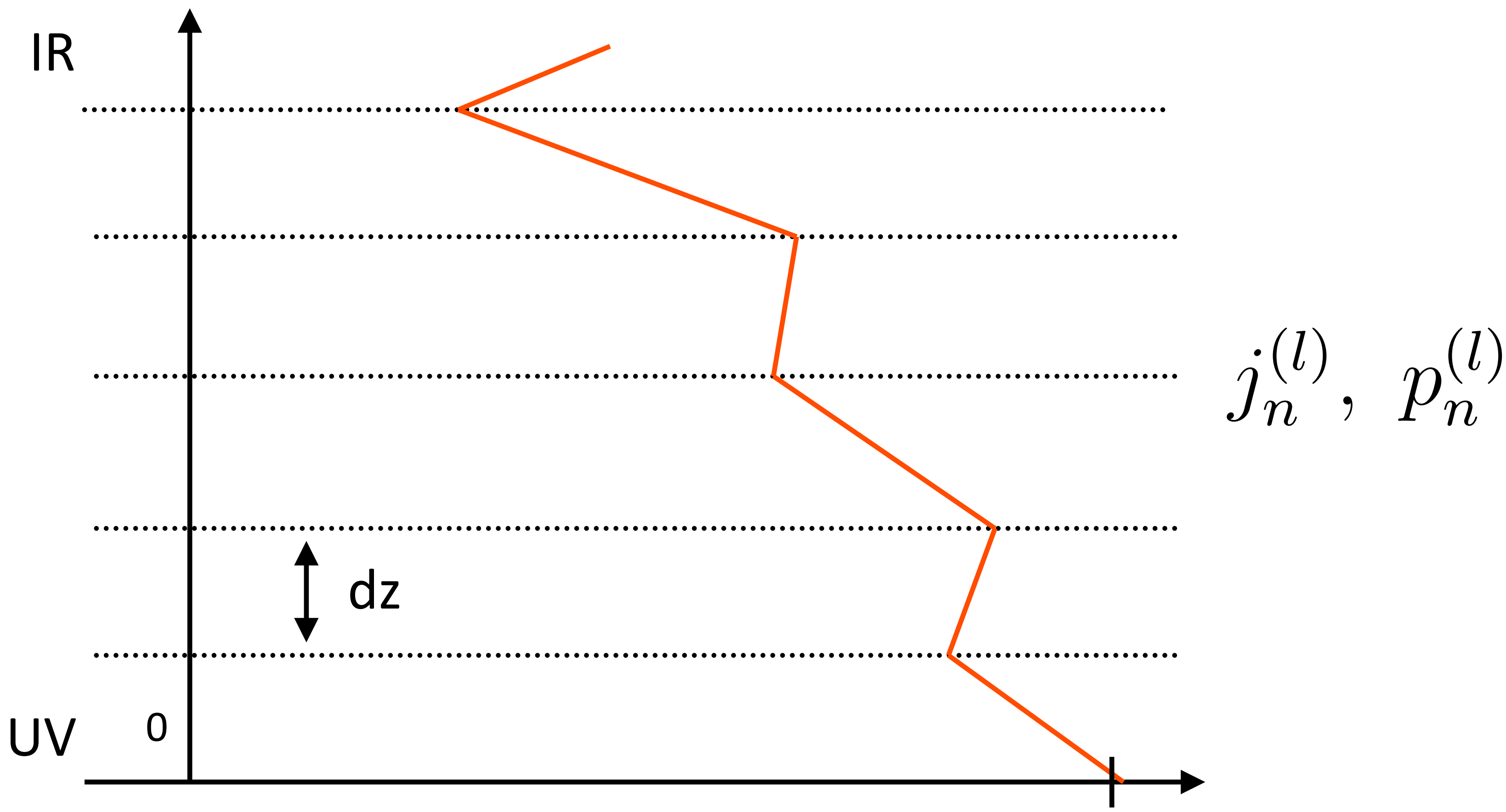
$$\mathcal{L} = J_{nm} p_n^{(1)} p_m^{(1)} + \sum_{i=1}^{\infty} \Big[p_n^{(i)} (j_n^{(i)} - j_n^{(i-1)}) + dz \mathcal{L}_c [j_n^{(i)}]$$

$$-dz A_n [j^{(i)}] p_n^{(i+1)} + dz B_{nm} [j^{(i)}] p_n^{(i+1)} p_m^{(i+1)} \Big]$$

$$j_n^{(0)}(x) = J_n(x)$$

- A set of dynamical sources and dynamical operators are introduced at each step of RG at the expense of decimating high energy mode bit by bit

Extra dimension as a length scale



Boundary condition $j_n^{(0)}(x) = J_n(x)$

Continuous extra dimension

$$j_n^{(l)}(x) \to j_n(x,z)$$
$$p_n^{(l)}(x) \to p_n(x,z)$$
$$z = ldz$$

$$Z[J(x)] = \int Dj_n(x,z) Dp_n(x,z)\ e^{iS_{D+1}}\Big|_{j(x,0)=J_n(x)}$$

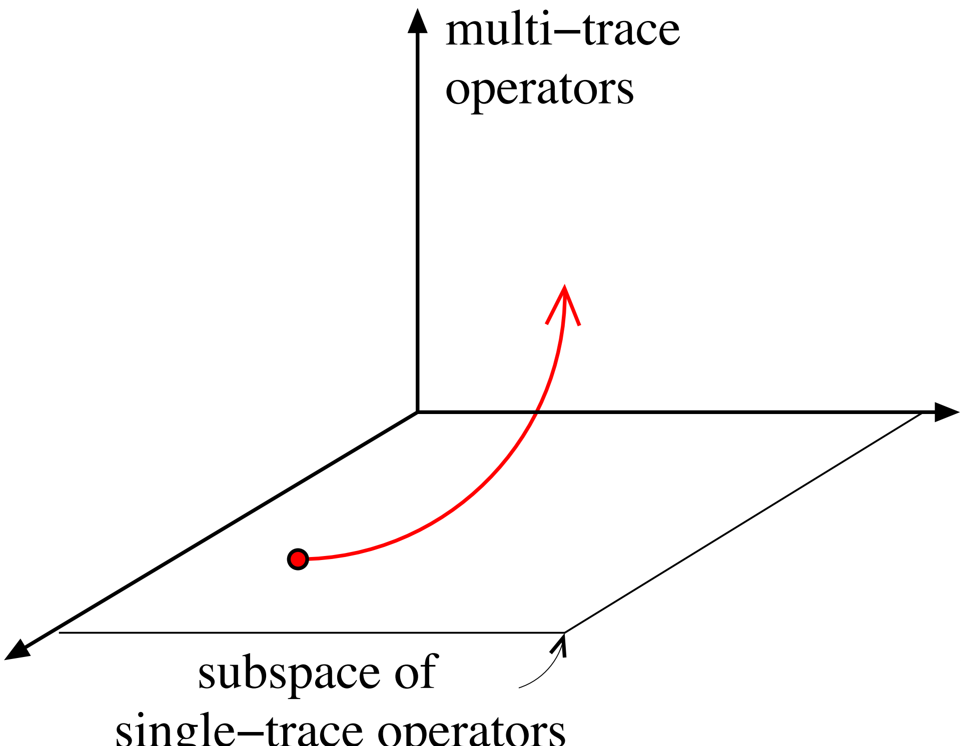
$$\boxed{\begin{aligned} S_{D+1} = \int dx J_{nm} p_n(x,0) p_m(x,0) + \int_0^\infty dz \int d^D x \Big[p_n(x,z)\partial_z j_n(x,z) \\ + \mathcal{L}_c[j_n(x,z)] - A_n[j(x,z)] p_n(x,z) + B_{nm}[j(x,z)] p_n(x,z) p_m(x,z) \Big] \end{aligned}}$$

- The length scale becomes an extra coordinate [Verlinde]

Key features

- An exact change of variables
- D-dimensional partition function can be written as (D +1)-dimensional functional integration for dynamical sources and operator fields
- General scheme : can be applied to any QFT
  - For general theory, the holographic description is not useful
  - In the large N limit, the holographic theory become classical
  - New types of quantum order associated with suppression of topological defects in the bulk [SL (2011)]
- Holographic theory always includes gravity
  - Energy momentum tensor ➔spin-2 sources

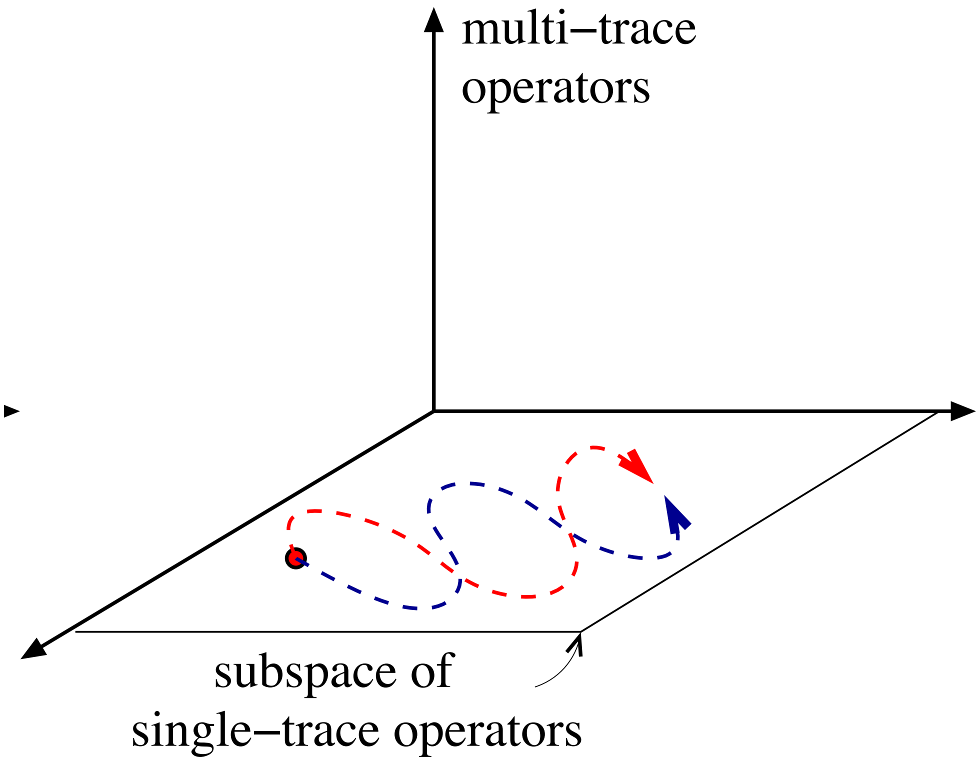
Conventional RG



$$\frac{dJ_{nm\ldots}}{d\ ln\Lambda} = \beta(J_n, J_{nm}, \ldots)$$

- Even though one starts with the fundamental operators at a given scale, non-fundamental operators are generated
- For a given initial condition, there is a unique trajectory = RG trajectory is classical (no fluctuations)

Holography



- Only fundamental operators appear
- The sources for fundamental operators become dynamical
- Quantum fluctuations in the RG trajectory : sources become operators!
- RG flow is governed by a quantum `Hamiltonian'
- In large N limit, saddle point path dominates : classical gravity

Quantum beta function

$$Z = \lim_{T \to \infty} < \Psi_f | e^{-iT\hat{H}} | \Psi_i >$$

Wavefunction for D -dimensional spacetime dependent sources

- It is useful to view the scale parameter z as `time'
- Dynamical sources and dynamical operators are conjugate to each other $[\hat{j}_l(x), \hat{p}_m(x')] = \delta_{l,m}\delta(x - x')$
- Partition function is written as a transition amplitude of D-dimensional quantum wavefunction of coupling constants
- The Hamiltonian generates scale transformation for dynamical couplings
- The Heisenberg equation : quantum beta function

$$\frac{d\hat{j}_n}{d\ z} = [\hat{H}, \hat{j}_n]$$

Local RG prescription

- Spacetime dependent coarse graining

$$-\phi\left(\nabla^2+\frac{\nabla^4}{\Lambda^2}+\ldots\right)\phi \to -\phi\left(\nabla^2+\frac{\nabla^4}{\Lambda(x)'^2}+\ldots\right)\phi$$

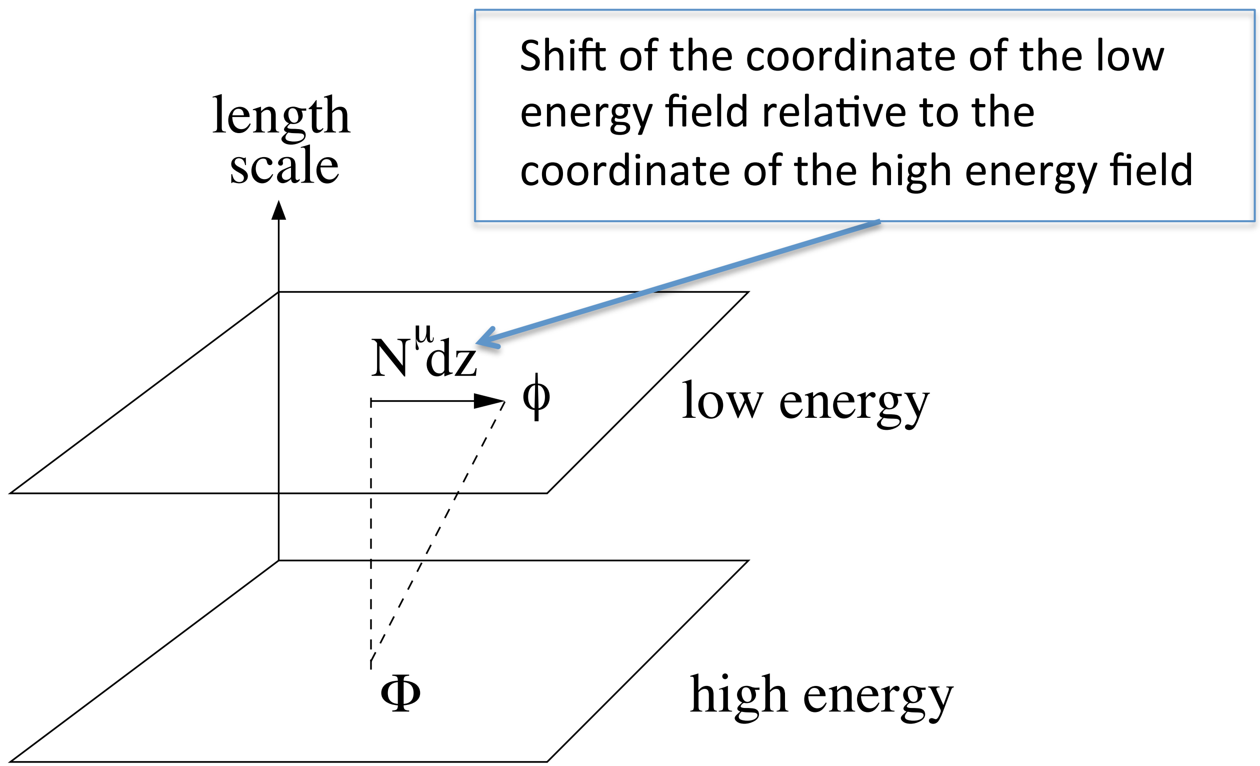
$$\Lambda(x)' = \Lambda e^{-\alpha(x)dz}$$

Speed of coarse graining

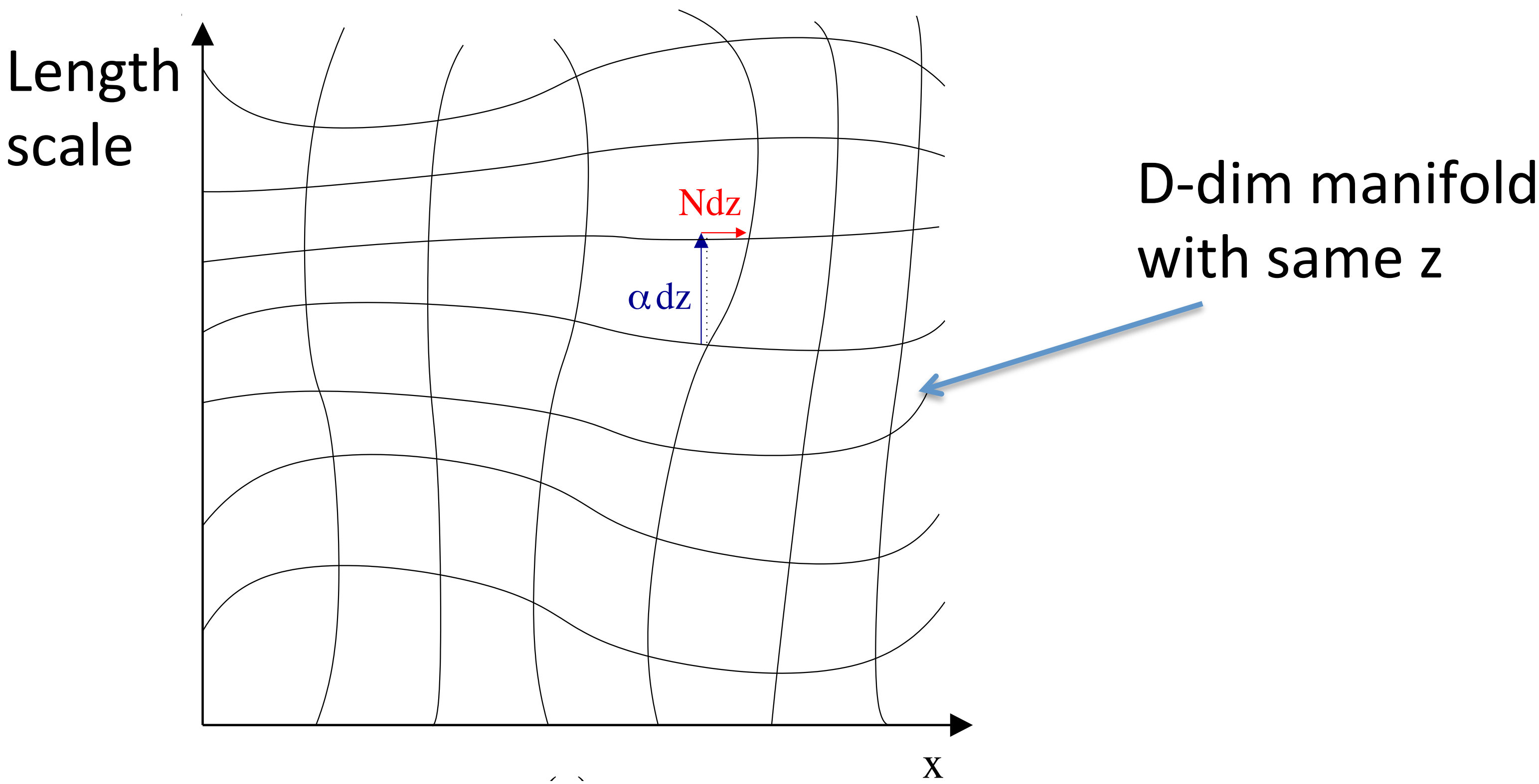
- By construction, Z is independent of α(x,z)
- Choosing different RG scheme α(x,z) : choosing different gauge

Shift

- One does not have to choose the coordinate of the low energy field as the coordinate of the high energy mode



Diffeomorphism = Freedom to choose different local RG schemes



- (D+1)-constraints are first-class

D-dimensional matrix field theory

single-trace operators

$$O_{[q+1;\{\mu_j^i\}]} = \frac{1}{N}\mathrm{tr}\left[\Phi\left(\partial_{\mu_1^1}\partial_{\mu_2^1}..\partial_{\mu_{p_1}^1}\Phi\right)\left(\partial_{\mu_1^2}\partial_{\mu_2^2}..\partial_{\mu_{p_2}^2}\Phi\right)\ldots\left(\partial_{\mu_1^q}\partial_{\mu_2^q}..\partial_{\mu_{p_q}^q}\Phi\right)\right]$$

Spacetime dependent sources

ϕ : N x N tracelss symmetric real matrix field

$$Z[\mathcal{J}] = \int D\Phi \;\; \exp\left[iN^2\int d^Dx\left(-\mathcal{J}^m O_m + V[O_m;\mathcal{J}^{\{m_i\},\{\nu_j^i\}}]\right)\right]$$

multi-trace deformation

$$V[O_m;\mathcal{J}^{\{m_i\},\{\nu_j^i\}}] = \sum_{q=1}^{\infty}\mathcal{J}^{\{m_i\},\{\nu_j^i\}}O_{m_1}\left(\partial_{\nu_1^1}..\partial_{\nu_{p_1}^1}O_{m_2}\right)\left(\partial_{\nu_1^2}..\partial_{\nu_{p_2}^2}O_{m_3}\right)\ldots\left(\partial_{\nu_1^q}..\partial_{\nu_{p_q}^q}O_{m_{q+1}}\right)$$

(D+1)-dimensional gravity

$$Z[\mathcal{J}] = \int \mathcal{D}J(x,z)\mathcal{D}\pi(x,z) \; e^{i\left(S_{UV}[\pi(x,0)]+S[J(x,z),\pi(x,z)]+S_{IR}[J(x,\infty)]\right)}\Big|_{J(x,0)=\mathcal{J}(x)}$$

Bulk action :
$$S = N^2 \int d^D x dz \left[(\partial_z J^n)\pi_n - \alpha(x,z)\mathcal{H} - N^\mu(x,z)\mathcal{H}_\mu\right]$$

Hamiltonian constraint :
$$\mathcal{H} = \tilde{A}^{\mu\nu}[J(x)]\pi_{[2,\mu\nu]} - \frac{\tilde{B}^{\mu\nu\lambda\sigma}[J(x)]}{\sqrt{|G|}}\pi_{[2,\mu\nu]}\pi_{[2,\lambda\sigma]}$$
$$-\sqrt{|G|}\left\{C_0[J(x)] + C_1[J(x)]\mathcal{R}\right\} + \ldots,$$

Momentum constraint :
$$\mathcal{H}_\mu = -2\nabla^\nu \pi_{[2,\mu\nu]} - \sum_{[q,\{\mu_j^i\}]\neq[2,\mu\nu]} \left[\sum_{a,b} \nabla_\nu \left(J^{[q,\{\mu_1^1\mu_2^1\ldots\mu_{b-1}^a \nu \mu_{b+1}^a\ldots\}]}\pi_{[q,\{\mu_1^1\mu_2^1\ldots\mu_{b-1}^a \mu \mu_{b+1}^a\ldots\}]}\right)\right.$$
$$\left. + (\nabla_\mu J^{[q,\{\mu_j^i\}]})\pi_{[q,\{\mu_j^i\}]}\right].$$

[SL (2012)]

Summary

- D-dimensional QFT can be explicitly mapped into a (D+1)-dimensional quantum theory of gravity based on a local RG
- Quantum beta function
- Example of emergent gravity
- Prove the Maldacena's conjecture ?
- Applications to concrete condensed matter systems ?