Transfer of BECs through Intrinsic Localized Modes (ILMs) in an Optical Lattice (OL)

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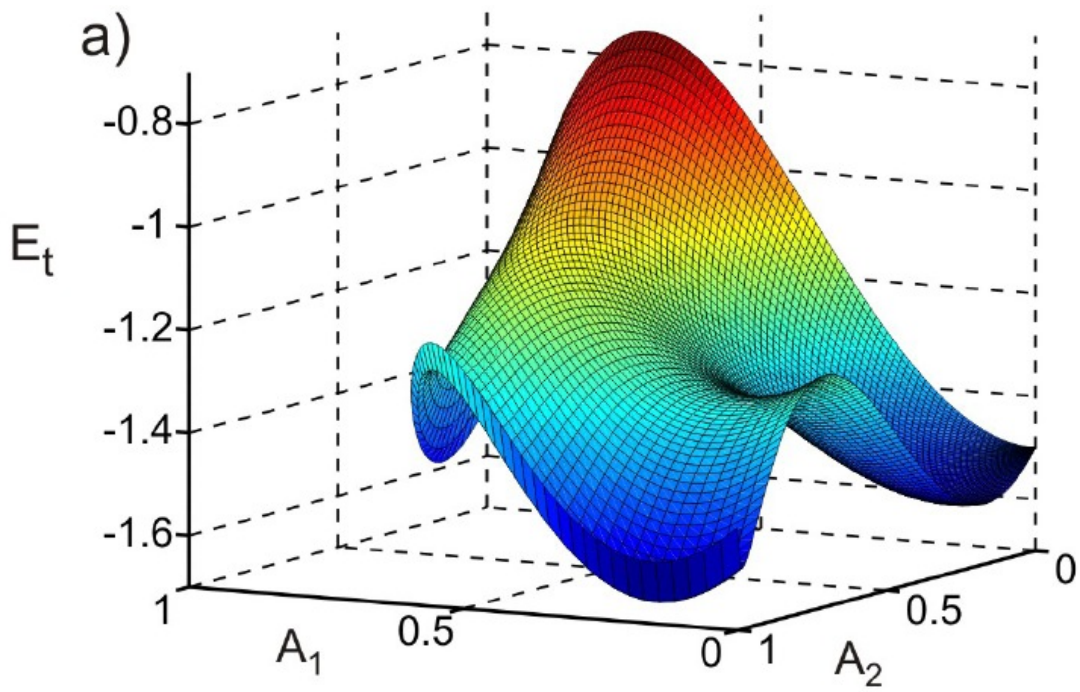


* With Holger Hennig (Harvard) and Jerome Dorignac (Montpellier)

Bottom Line

- Previous numerical and experimental results have shown that leakage of Bose Einstein Condensates from extended Optical Lattices can be controlled and manipulated by Intrinsic Localized Modes formed in (one of) the wells of the Optical Lattice.
- We show that a heuristic (and analytic) understanding of these results on extended lattices follows from studying the Peierls-Nabarro energy landscape of a *three-well* Optical Lattice (nonlinear trimer).

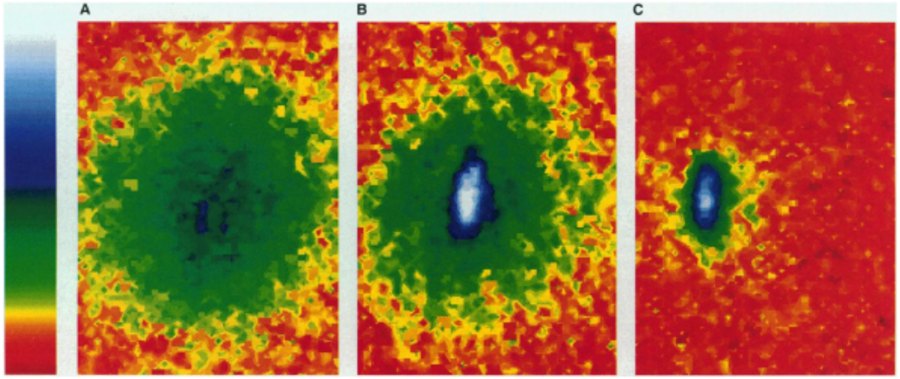
Outline

- "Bose-Einstein Condensates" (BECs)
  - Concept and Images
  - Gross-Pitaevskii Equation (GPE)—solitons in 1D?
- BECs in "Optical Lattices" (OLs)
  - Concept and images
  - Discrete Nonlinear Schrödinger Equation (DNLSE)
- Interlude on "Intrinsic Localized Modes" = (ILMs)
- Excitations of BECs in OLs and their interactions
  - Previous results on transfer ("leakage") of BECs from OLs
- A Heuristic Understanding of ILM Interactions
  - The Nonlinear Trimer as a Model System
  - The Peierls-Nabarro Barrier and Energy Landscape
- Results for Extended Systems
- Summary and Conclusions
- Reference: *Physical Review A* **82** 053604 (2010)

What is a BEC ?

- Bose-Einstein (BE) statistics describe indistinguishable quantum particles, differ from classical (Boltzmann) statistics and are known to apply to integer spin quantum particles (phonons, atoms, molecules), which are now called "bosons.".
- Most striking difference in BE statistics is that any number of bosons can occupy a given quantum state (no "exclusion")
- In 1925 Einstein showed that this implies that at very low temperatures bosons "condense" into the lowest quantum state, creating a quantum system with (potentially) a macroscopic number of particles—a Bose-Einstein condensate (BEC).
- In 1995, group of Carl Wieman and Eric Cornell at JILA observed BEC in ultracold gas of Rb^{87} atoms; a few months later, group of Wolfgang Ketterle observed a much larger number of atoms in a BEC in Na^{23}. The three shared the 2001 Nobel Prize in Physics for their discoveries.

First Image of a BEC



False color images of the velocity distribution of a cloud of Rb^{87} atoms near T_C = 170 nk. a) above transition; b) emerging condensate; c) well-developed condensate of roughly 2000 atoms, described by a macroscopic quantum wave function, Φ

E. Cornell and C. Wieman *Science* **269**, 198-201 (1995)

Modeling BECs

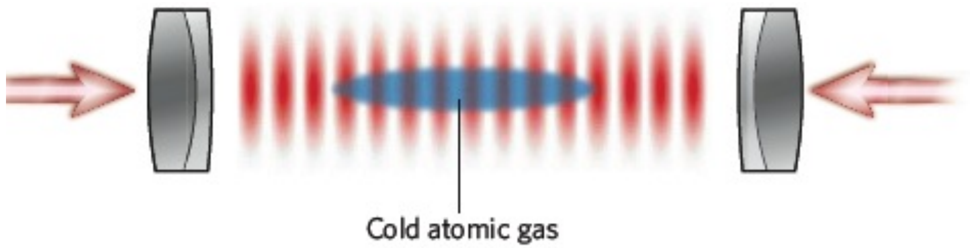
BEC is described by a quantum wavefunction Φ(x,t). In the context of cold atomic vapors trapped in an external potential (typically combined magnetic confinement and optical potential)

Φ(x,t) obeys the semi-classical "Gross-Pitaevskii" equation (GPE)

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + [V_{\mathrm{ext}} + g_0|\Phi|^2]\Phi$$

- Nonlinear dynamicists recognize this immediately as a variant of the Nonlinear Schrödinger equation, here in 3D and with an external potential—more on this later.
- The interaction term—g_0—describes s-wave scattering of atoms
- In 1D case (realizable depending on external potential) we expect (continuum) solitons among excitations.

BECs in an Optical Lattice



Counter-propagating laser pulses creating standing wave that interacts with atoms, so that the BEC experiences a periodic potential of the form:

$$V_{\text{ext}}(\vec{r}) = U_L(x,y)\sin^2[2\pi z/\lambda]$$

$U_L(x,y)$ is transverse confining potential, λ is the laser wavelength (typically 850 nm) and "z" is the direction of motion.

Image from I. Bloch *Nature* **453** 1016-1022 (2008)

Modeling BECs in Optical Lattices

- Quantum system in periodic potential—just as in solid state. Can describe in terms of (extended) Bloch wave functions or (localized) Wannier wave functions, centered on the wells of the potential (but allowing tunneling between the wells).
- Expanding the GPE in terms of Wannier functions leads to the Discrete Nonlinear Schrodinger Equation (DNLSE) for the wave function amplitudes. In normalization we shall later use

$$i\frac{\partial \psi_n}{\partial t} = \lambda|\psi_n|^2\psi_n - \frac{1}{2}[\psi_{n-1} + \psi_{n+1}]$$

- Comes from Hamiltonian ($\lambda=2U/J$)

$$\mathcal{H} = \sum_{n=1}^{M}[U|\psi_n|^4 + \mu_n|\psi_n|^2] - \frac{J}{2}\sum_{n=1}^{M-1}(\psi_n^*\psi_{n+1} + c.c.)$$

Brief Interlude on “ILMs”

- Definition: an “intrinsic localized mode” (ILM)—is a spatially localized, time-periodic, stable (or at least very long-lived) excitation in a *spatially extended, perfectly periodic, discrete system*. Also known as a “discrete breather” (DB).
- Bottom Line: The mechanism that permits the existence of ILMs has been understood theoretically for nearly two decades. Only in last decade have they been observed in physical systems as distinct as charge-transfer solids, Josephson junctions, photonic structures, micromechanical oscillator arrays, and BECs trapped in optical lattices.

Specific Example: Φ^4 Equation

Specific model for infinite chain of coupled oscillators

$$\frac{d^2\phi_n}{dt^2} - \quad \phi_n + \phi_n^3 = 0.$$

- Coupling ~ $1/(\Delta x)^2$
- "Quartic" double-well oscillator $\phi_n(t)$ at each site (n) of infinite "lattice":minimum at $\phi_n = 1$

Key point

- Spectrum of *linear* oscillations about minimum $\omega_q^2 = 2 + (2/\Delta x)^2 \sin^2(q/2)$ is a band, bounded from above and below— *upper cut-off from discreteness.*

ILMs: Intuition and Theory

- For nonlinear oscillations about minimum in quartic potential, frequencies decrease with amplitude (like plane pendulum) so one can create a *localized, nonlinear mode (ILM)* with frequency ω_b *below* linear spectrum.

- If Δx is large enough so that coupling ~ $1/(\Delta x)^2$ is small, the band of excitations $\omega^2_q = 2 + (2/\Delta x)^2 \sin^2(q/2)$ is very narrow, so that the second harmonic of ω_b can lie above the top of the band. Thus there can be not (linear) coupling of local mode (ILM) to extended states and the ILM is linearly stable.

- Key intuition on stability of ILMs: discreteness produces finite band structure for linear excitations and nonlinearity means that frequencies of ILMs can be tuned to lie outside of band (and multiples).

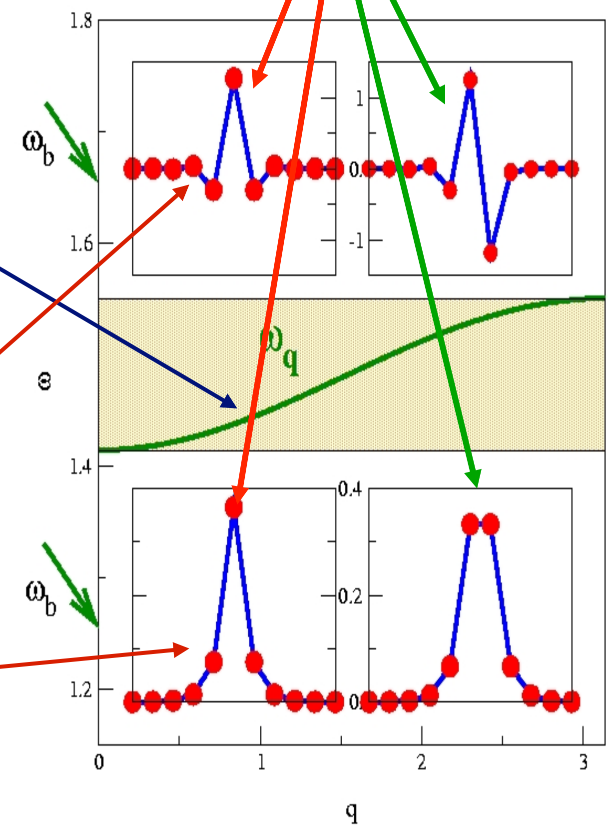
Explicit Example of ILMs in Φ^4

Figure shows linear band (grayish yellow) with energy-momentum dispersion relation

$$\omega_q^2 = 2 + (2/\Delta x)^2 \sin^2(q/2)$$

for Δx =10. The isolated localized mode frequencies, ω_b, are shown for the types of ILMs shown in the top and bottom panels. Note that ILMs can occur both *above* and *below* the linear band—those above have an optical character (adjacent particles out of phase), whereas those below have an acoustic character (adjacent particles in phase). There are many ILMs—only four are shown here.

- Both on-site and "between"-site ILMs



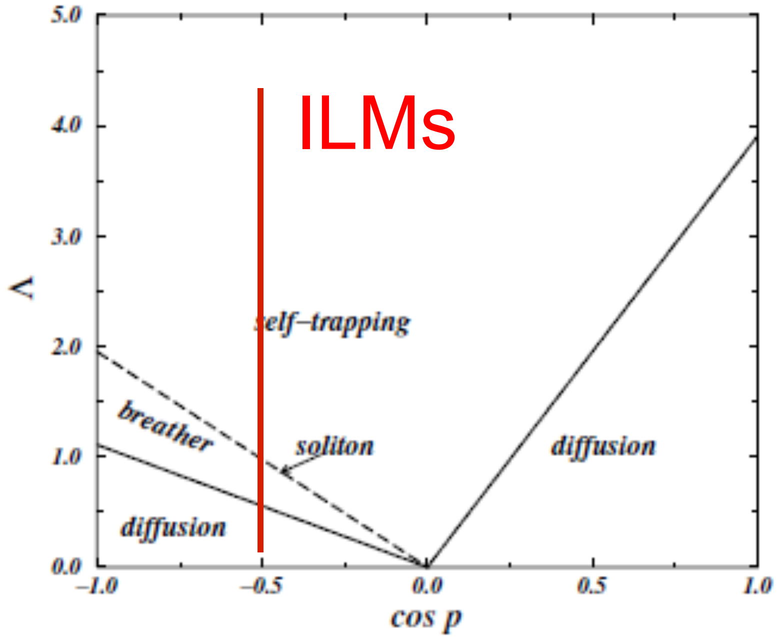
Where do we stand?

- After interlude on ILMs, let's recap what we know.
  - BECs in 1D OLs are described by the DNLSE

$$i\frac{\partial \psi_n}{\partial t} = \lambda|\psi_n|^2\psi_n - \frac{1}{2}[\psi_{n-1} + \psi_{n+1}]$$

  - Based on discussion of ILMs, we expect them to occur—along with linear waves and possibly other excitations—in the BEC in OL.
  - Is this expectation confirmed? If so, what are properties of BEC ILMs and other excitations?

Excitations of BECs in OLs



Notice that there exist many kinds of excitations, depending on parameters—eg, for *cos p*=-0.5, as Λ ~λN is increased, one sees linear excitations ($\omega \sim k^2$), then localized moving breathers that become solitons along a single line in the plane, then "self-trapped states" = ILMs.

Fig. 2 from A. Trombettoni and A. Smerzi, *PRL* **86**, 2353 (2001) : variational collective coordinate Ansatz: *cos* p related to momentum of variational solution

Excitations of BECs in OLs and their interactions

Results of Ng et al (Geisel group) confirm for extended systems (M ~ 300) that ILMs are not formed at small Λ (=λN/M) but are readily formed at large Λ.

Note that other excitations (linear waves and traveling breathers) are typically "trapped" between the ILMs, although sometimes they appear to pass through.

This is consistent with results in other ILM-bearing systems: at low incident energy of interaction, ILM reflects other excitation but at higher energy, at least some of amplitude can be transmitted: How can we understand this?

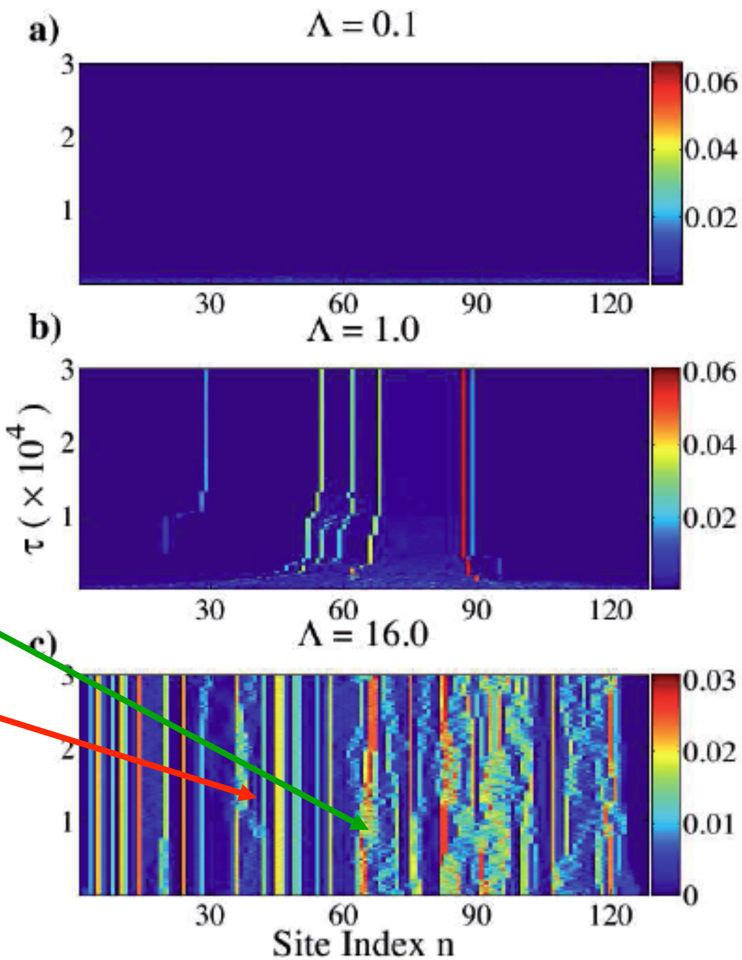
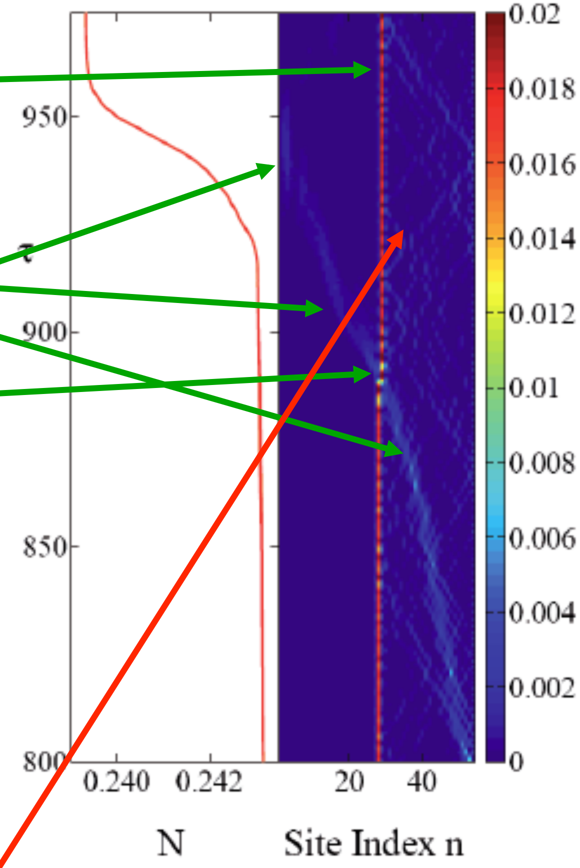


Fig 2 from Ng *et al.*, *New J. Phys* **11**, 073045 (2009).

Excitations of BECs in OLs and their interactions

An event in which a single ILM interacts with a number of moving excitations (right to left) and some amplitude is transmitted through the ILM to the edge of the lattice at N=0. Note that the ILM moves (a single site) in the direction of the incoming excitation. The arrival of the amplitude at n=1 leads to a leaking of the condensate and is called an “avalanche” by the authors. Scattering from ILMs controls the avalanches:
How can we understand this ?



NB: Low energy excitations reflect off ILM

Fig 5 from Ng *et al.*, *New J. Phys* **11**, 073045 (2009).

Summary of Transport through ILM

- Results of Ng *et al.* include
  - Little transport through ILM at low energy, excitations reflected
  - Clear threshold energy, E_{thrs}, for significant transport through ILM
  - Above threshold, ILM *moves one lattice site* in direction of incoming excitation
  - Transport and ILM motion linked to leakage avalanches of BEC from OL
- These results obtained by direct numerical simulation of DNLSE on extended lattices. In remainder of talk, we will give heuristic, mostly analytic explanation of the results.

Heuristic Understanding of ILM Interactions

- ILMs are highly localized and are “self-trapped.” Their properties should follow from small systems.
- Smallest system that allows ILM plus perturbations *and* transport is trimer—already established as useful for studying ILMs.
- DNSLE for trimer (open BCs) becomes

$$
\begin{aligned}
i\partial_t\psi_1 &= \lambda|\psi_1|^2\psi_1 - \frac{1}{2}\psi_2 \\
i\partial_t\psi_2 &= \lambda|\psi_2|^2\psi_2 - \frac{1}{2}(\psi_1 + \psi_3) \\
i\partial_t\psi_3 &= \lambda|\psi_3|^2\psi_3 - \frac{1}{2}\psi_2 \,.
\end{aligned}
\qquad
N = \sum_{n=1}^{M} |\psi_n|^2 = 1 \,.
$$

Analytic approach to DNLSE for Trimer

Assuming $\psi_n(t) = A_n e^{iwt}$ leads to

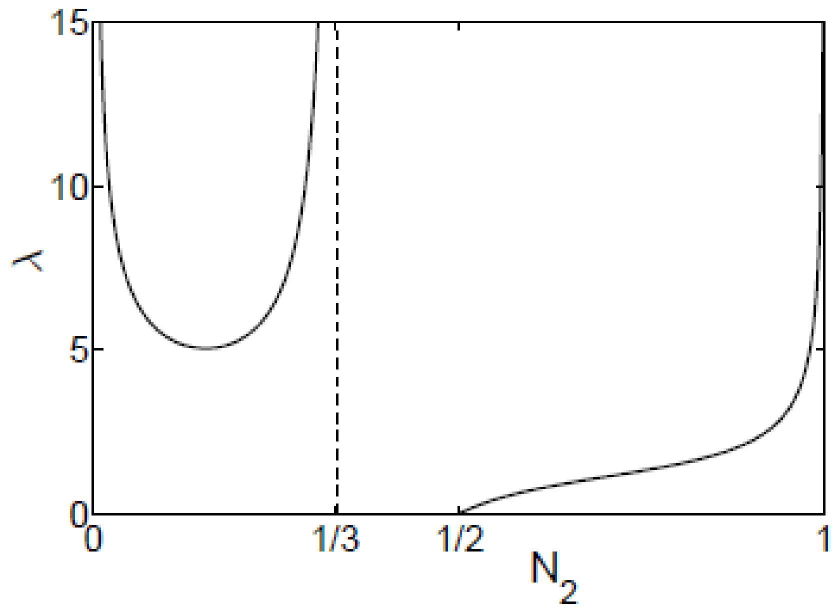
$$\begin{aligned} -wA_1 &= \lambda A_1^3 - \frac{1}{2}A_2 \\ -wA_2 &= \lambda A_2^3 - A_1 \\ 1 &= 2A_1^2 + A_2^2\,. \end{aligned}$$

$N_i = A_i^2$

$$\lambda(N_2) = \frac{\sqrt{2}(2N_2 - 1)}{\sqrt{N_2(1 - N_2)}(3N_2 - 1)}$$

Only one solution for small λ.

Three solutions for λ> ~5.04



ILM solutions and heuristic approach

In limit of infinite λ, three types of (symmetrical) solutions are

1) "bright" ILM ,
2) "dark" ILM; and
3) phase-wise and anti-phase wise time-dependent moving solutions

$$\vec{\psi}_{(x3)}(0) = (0, 1, 0)$$

$$\vec{\psi}_{(x4)}(0) = (\sqrt{1/2}, 0, \sqrt{1/2})$$

$$\vec{\psi}_{(x5,x6)}(0) = (1/\sqrt{3}, \mp 1/\sqrt{3}, 1/\sqrt{3})$$

Focus on bright ILM solution, perturbed by "excitation" at site 1, study "dynamics" not only via direct integration of DNLSE (trivial) but also by energy considerations based on "Peierls-Nabarro" barrier and PN energy landscape concepts.

Recall "Peierls-Nabarro" Barrier

Consider a highly localized ILM. As we have seen, it can be centered at a site (a) or between sites (b).

A priori, there is no reason that the energies associated with these two positions should be the same. Indeed, Peierls and Nabarro showed that in general they were not, and the "PN barrier" is E_c-E_b, where E_c is the energy of a ILM centered at a site and E_b is energy between two lattice sites. PN barrier may be viewed as minimum barrier to translating the "self-trapped" ILM by a single lattice period.

Peierls-Nabarro Energy Landscape

Definition of PN Energy Landscape: For given A_n , extremize the full Hamiltonian H wrt phase differences: the minimum is the lower part of PN landscape, maximum is the upper part.

$$H = \frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) - (A_1 A_2 \cos(\phi_1 - \phi_2) + A_2 A_3 \cos(\phi_2 - \phi_3))$$

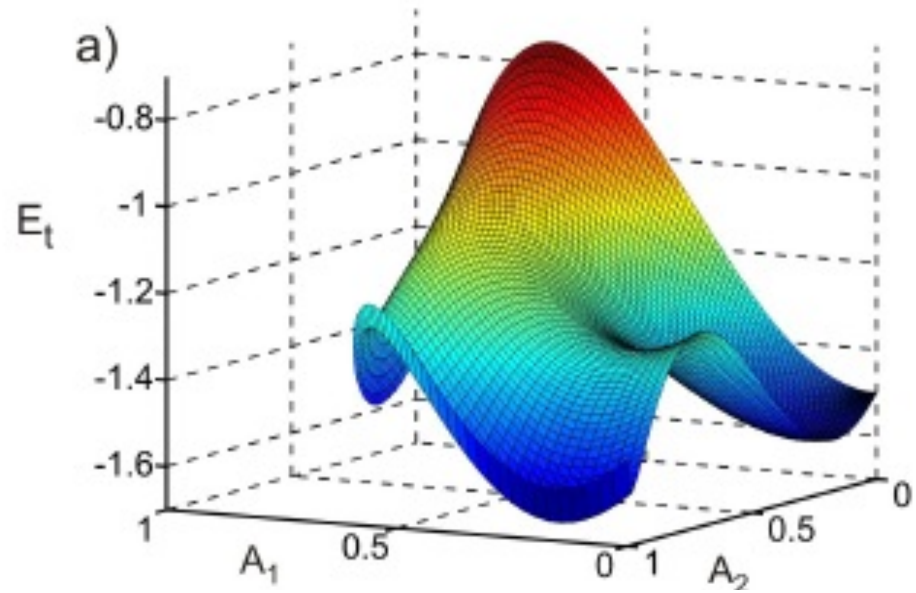
$$H_{\rm PN}^l = \min_{\delta\phi_{ij}}(-H) \quad ; \quad H_{pn}^u = \max_{\delta\phi_{ij}}(-H)$$

$$H_{\rm PN}^l = -\frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) - (A_1 + A_3)A_2$$

$$H_{\rm PN}^u = -\frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) + (A_1 + A_3)A_2 \ .$$

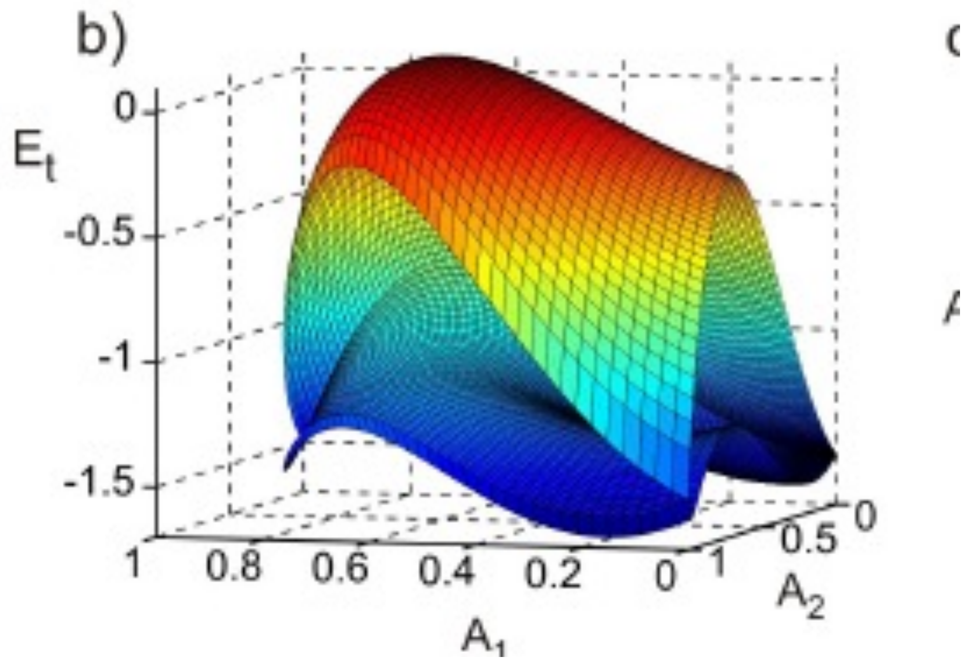
Images of PN Energy Landscapes (next slides)

Peierls-Nabarro Energy Landscape



The lower “sheet” of the PN Energy Landscape for λ = 3; there are three minima separated by saddle points.

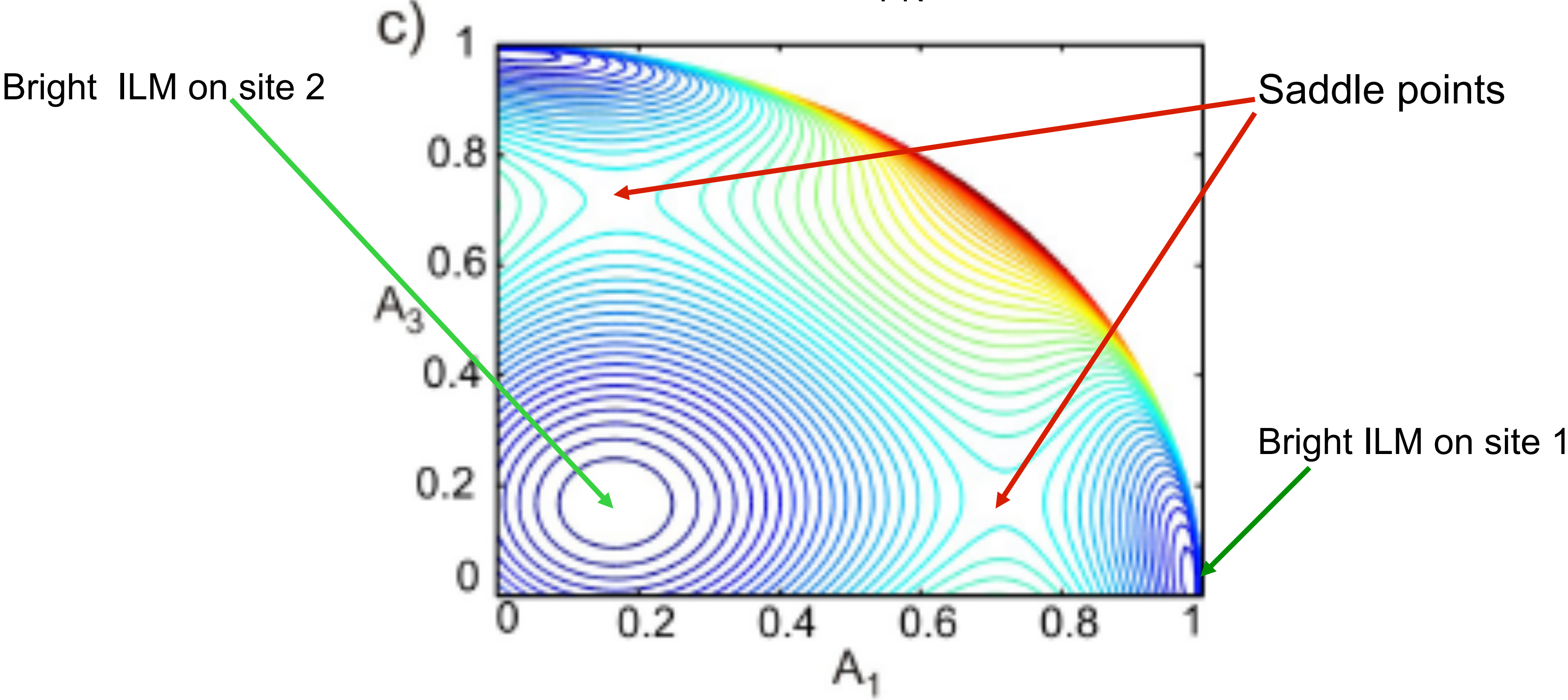
Peierls-Nabarro Energy Landscape



The phase space of the trimer is restricted by the PN “shell” which exists between the lower and upper sheets for PN Energy Landscape (figure for λ=3).

Peierls-Nabarro Energy Landscape

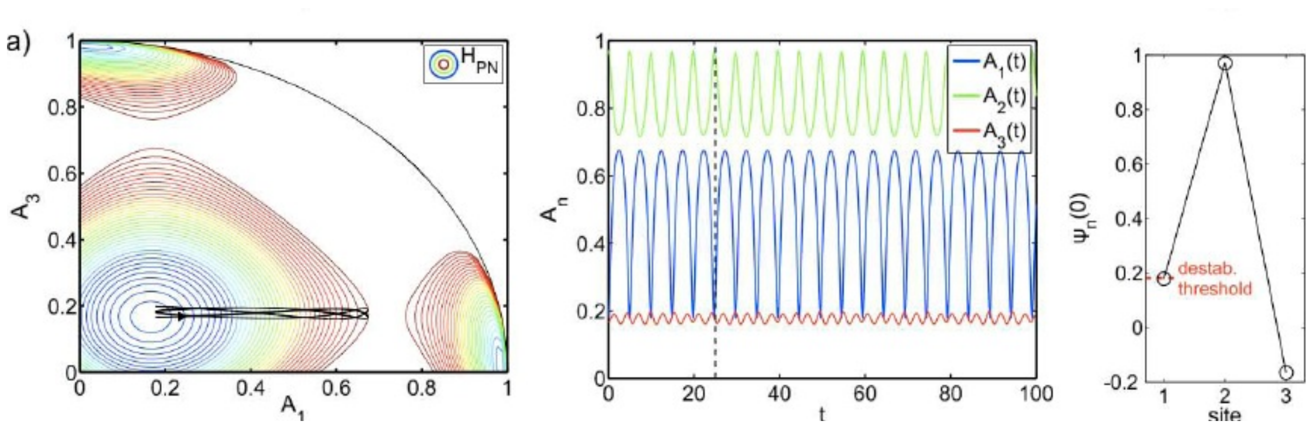
Contour plot of H^{I}_{PN} for λ=3

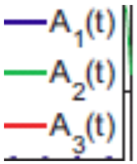


Next 4 slides show dynamics after initial perturbation of form

$$\vec{\psi}(0) = (-(A_1^b + \delta_A)e^{i\delta_\phi}, A_2, -A_1^b),$$

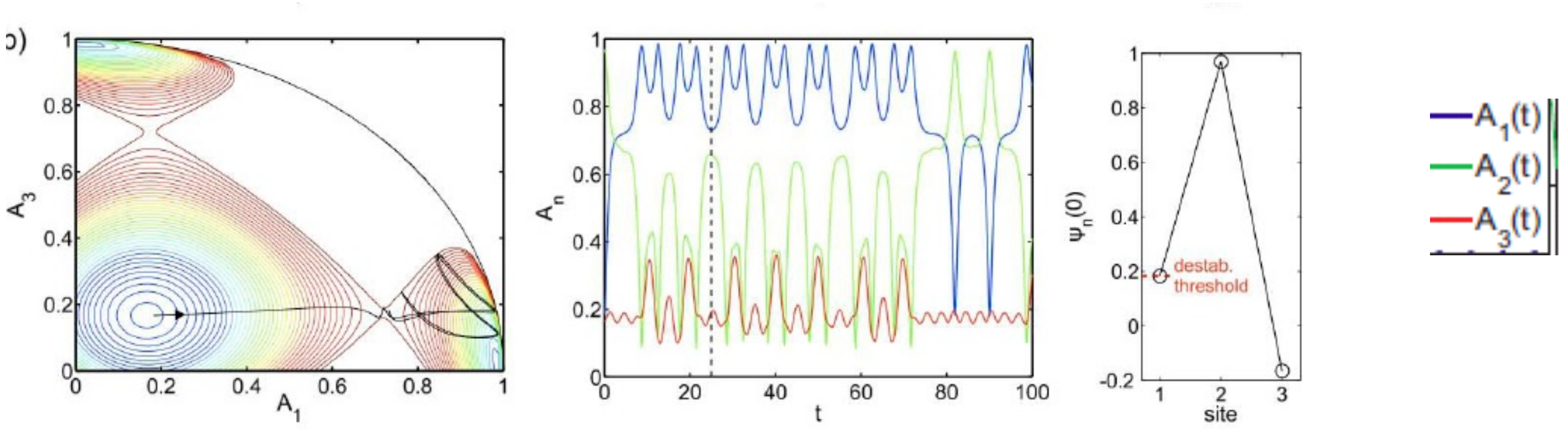
Dynamics on the PN Landscape





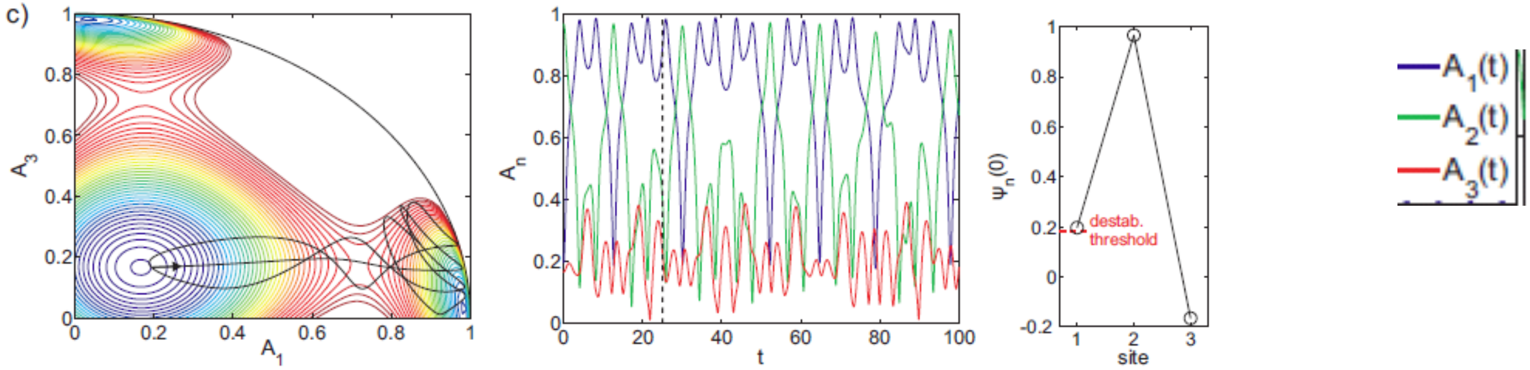
The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for . Note that the total energy E_t is below the threshold ~ height of saddle point so that the motion is confined to a “region” around the bright breather. The solution is plotted for 25 units (see vertical dashed line). The right panel shows that the energy of the initial condition (E_t=-1.32) is below E_{thrs} (-1.31).

Dynamics on the PN Landscape



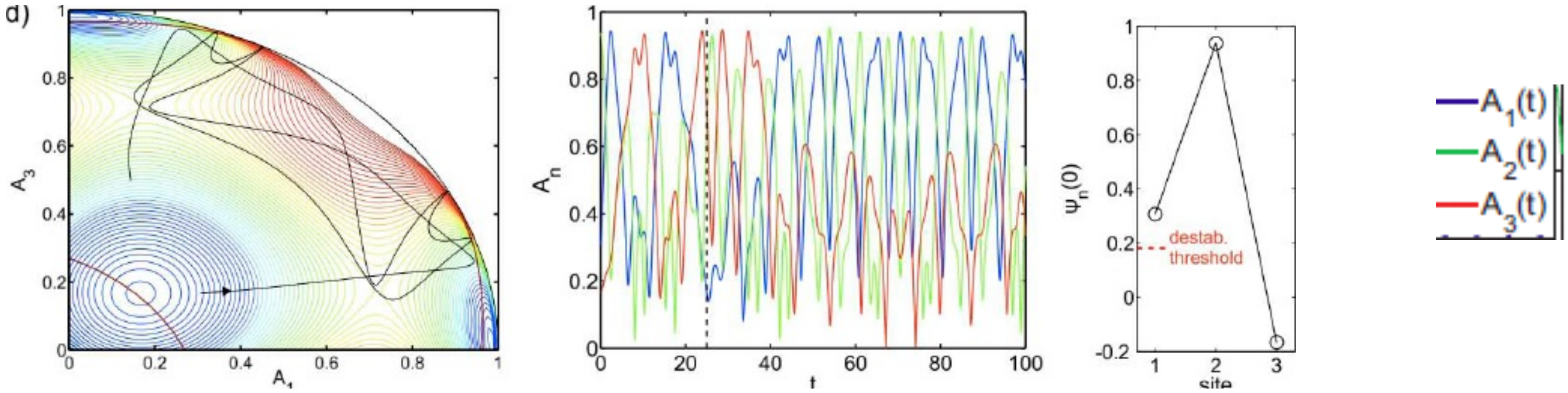
The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for $E_t=E_{thrs}=-1.31$; we see that the rim of the PN landscape clearly restricts the dynamics and governs the destabilization of the ILM. The size of right “lobe” gives the maximum of A_3, the amplitude transferred to site 3.

Dynamics on the PN Landscape



The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for E_t=-1.28. Note that the bottleneck at the rim widens and the amplitude transferred to site 3 is increased.

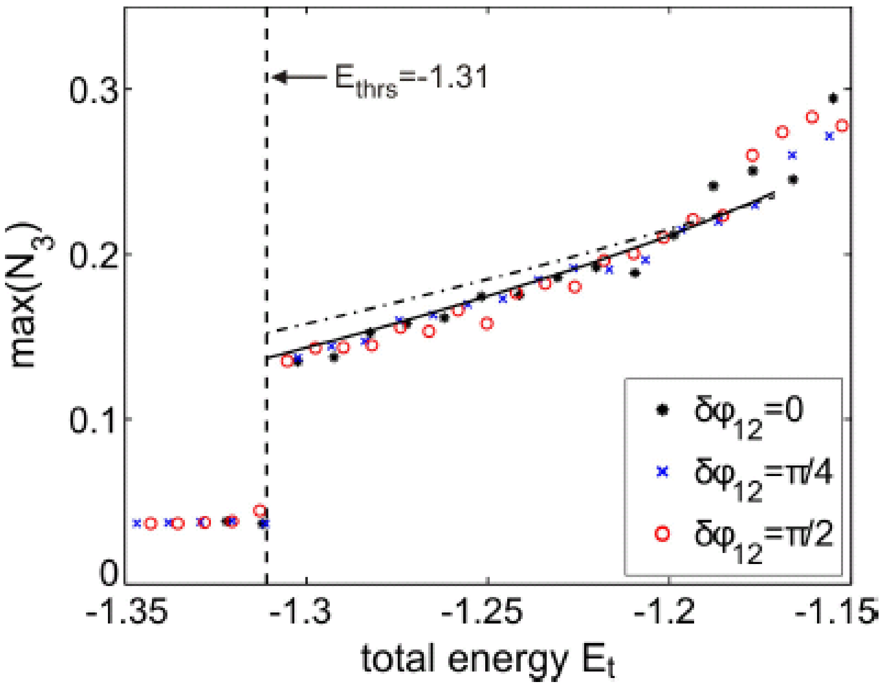
Dynamics on the PN Landscape



The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for E_t=-1.04, well above E_{thrs}. The motion explores large parts of the phase space and visits all three sites. The red line indicates where the upper PN energy landscape limits the motion; note that it prohibits A_n values close to 1.

Norm Transfer vs. Total Energy

Maximum value of the norm at site 3 after a collision of a ILM at site 2 with a lattice excitation "coming" from site 1 as a function of total energy on the trimer. For $E_t < E_{thrs} \sim -1.31$ (for $\lambda=3$) the ILM is stable and almost no transfer occurs. For $E > E_{thrs}$, ILM is unstable, moves to site 1, and the transmitted amplitude rises sharply (and is independent of the phase difference).

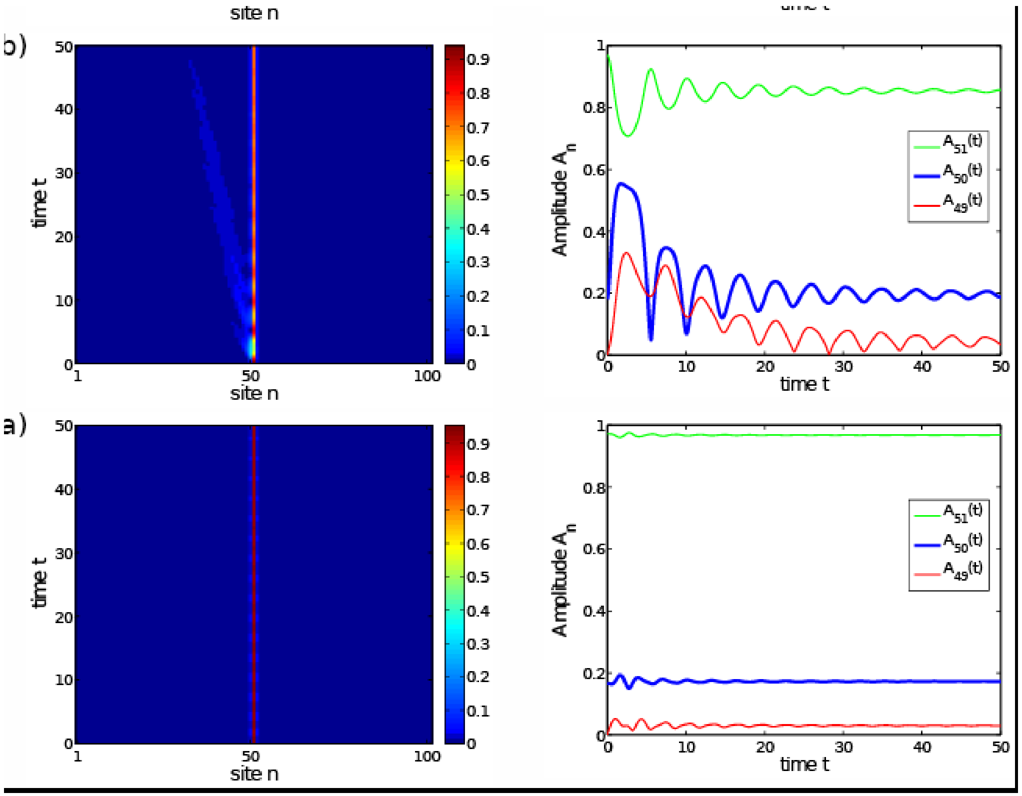


Solid line=numerics. Dashed line = analytic result from PN energy landscape calculation.

Results on Extended Optical Lattices

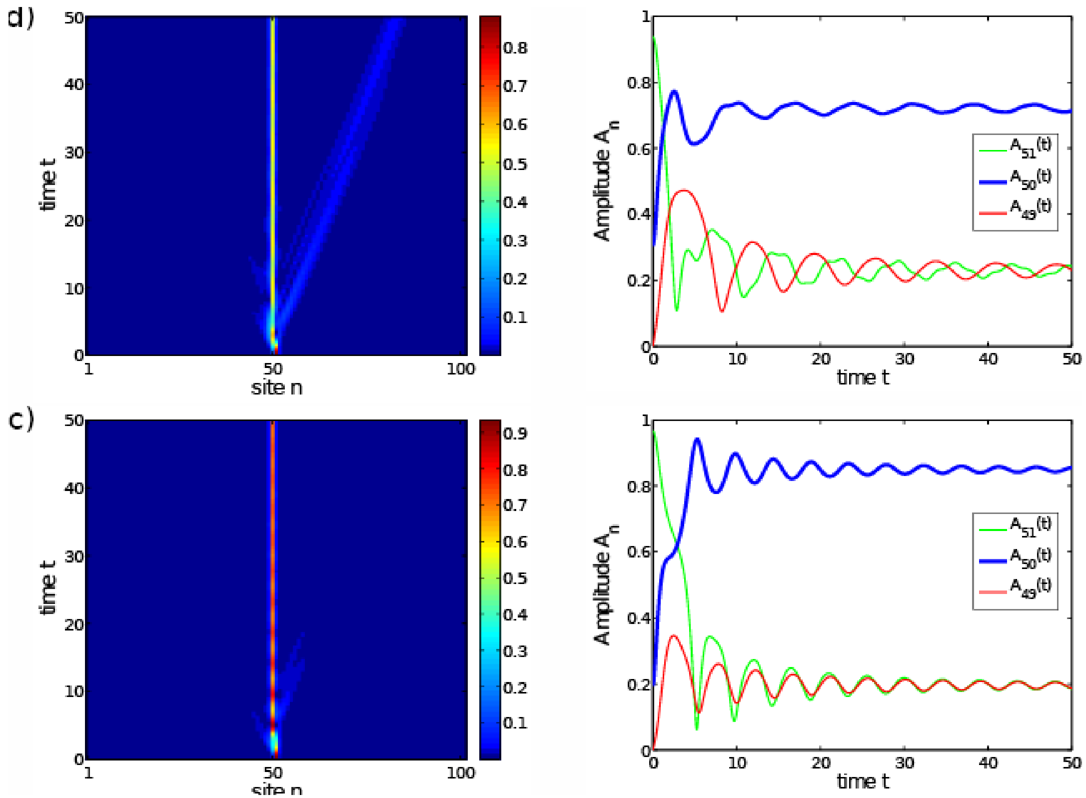
To test mechanism on extended lattices, we study the case N=101 with a ILM of the form studied in the nonlinear trimer located at the center of the lattice (N_0 =51). In the next two slides, we show the results of increasing the total energy on the trimer. We find that in the extended system, the energy threshold for destabilization and consequent motion of the ILM is slightly *higher* than in the trimer—we attribute this to the existence of many other degrees of freedom in the extended system. However, at energies very close to the threshold predicted from the trimer (E_{thrs}), the destabilization and migration of the ILM do occur, confirming that the mechanism that we have described analytically in the trimer does capture the behavior of the extended system.

Results at low energy



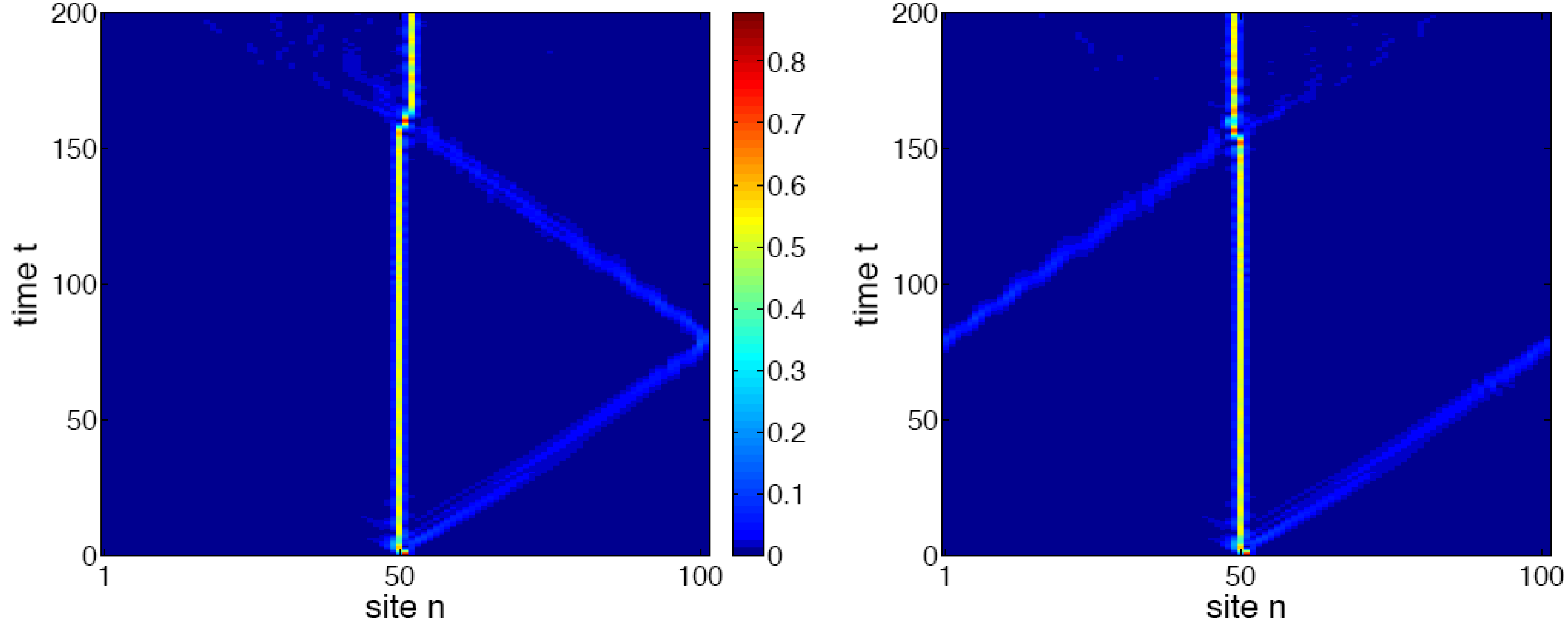
N=101 lattice. a) unperturbed “trimer” ILM at site 51; b) Perturbation on site 50 is just above trimer threshold. E_{thrs}; the ILM on extended lattice remains (barely) stable and the energy is reflected off to left as a low amplitude, seemingly localized excitation. The right panels confirm that in neither case does the ILM move.

Results at higher energy



N=101 lattice. c) ILM migrates by one site towards the perturbation after t ~ 5 time steps; d) ILM migrates at t~ 2 time steps and a localized excitation (moving breather) moves off to right, indicating the passage of energy through the ILM. Note that panels on right confirm that ILM has moved.

Results for Longer Times



N=101 lattice. This shows results for longer time. Left panel shows reflection of moving breather off the open boundary and then second interaction with ILM leading to two site migration. Right panel shows periodic boundary conditions (so discrete pseudomomentum is conserved.) Note that in each case, ILM moves in direction of incoming wave, as predicted by PN analysis of trimer.

Summary and Conclusions

- The threshold for collisions leading to “tunneling” of amplitude through ILM is sharp and is related to the loss of stability of the ILM.
- The loss of stability of the ILM is related to the PN barrier in which the ILM is self-trapped.
- When amplitude is transmitted, the ILM moves by one site in the direction of the incoming wave.
- The dynamics of the interactions of moving excitations with ILMs, including the size of the transmitted amplitude and the motion of the ILM, can be understood in terms of motion on the fixed energy contours of a PN Energy Landscape.
- The nonlinear trimer provides a quantitative fit to the results observed in extended systems and allows us to obtain an analytic expression for the transmitted amplitude.

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QUESTIONS ?

Leakage via Avalanches

Extensive studies of OLs with differing numbers of wells (128<M<1024) and different values of Λ. The authors establish a scaling law showing that the avalanche size $P(J) \sim J^{-1.86}$. Scattering from DBs controls the avalanches: How can we understand this?

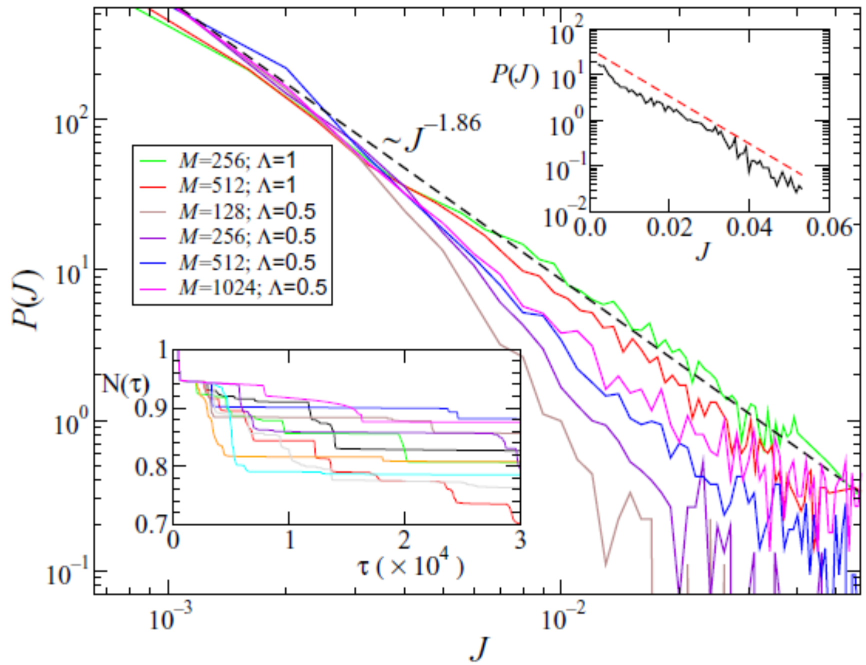
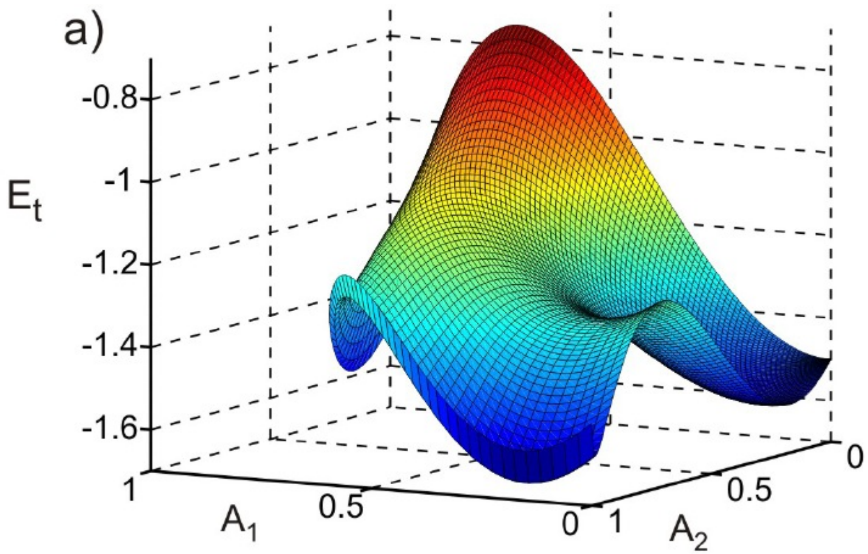


Fig 4 from Ng *et al.*, *New J. Phys* **11**, 073045 (2009).

Physics, Music, and Life: Four Decades of Freude und Freundschaft With Dionys Baeriswyl

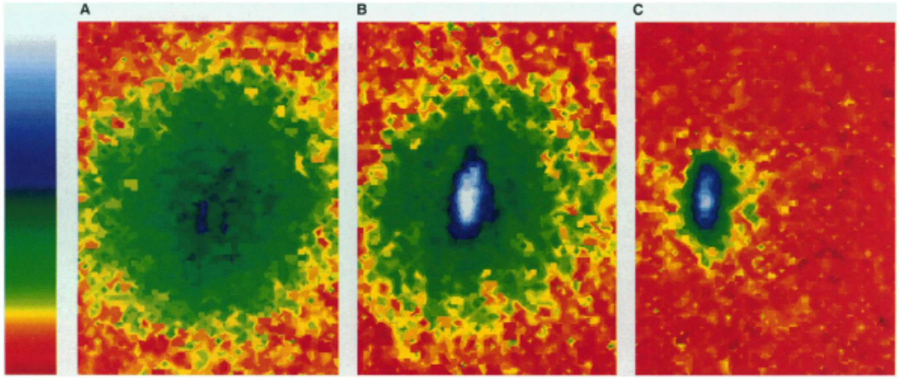
David K. Campbell
Boston University
Fribourg, April 15 2011



Bottom Lines

- Previous numerical and experimental results have shown that leakage of BECs from extended OLs can be controlled and manipulated by (ILM)s (formed in (one or more of) the wells of the OL.
- Increased transmission through ILM (and consequent leakage) is consequence of destabilization (and subsequent motion) of ILM by incoming wave.
- We show that a heuristic (and analytic) understanding of these results on extended lattices follows from studying the Peierls-Nabarro energy landscape of a *three-well* OL (nonlinear trimer).

First Image of a BEC



False color images of the velocity distribution of a cloud of Rb^{87} atoms near T_C = 170 nk. a) above transition; b) emerging condensate; 3) well-developed condensate of roughly 2000 atoms, described by a macroscopic quantum wave function, Φ

E. Cornell and C. Wieman *Science* **269**, 198-201 (1995)

Modeling BEC

- BEC is described by a quantum wavefunction Φ(x,t). For cold atomic vapors trapped in an external potential (combined magnetic confinement and optical potential) Φ(x,t) obeys the semi-classical “Gross-Pitaevskii” equation (GPE)

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + [V_{\mathrm{ext}} + g_0|\Phi|^2]\Phi$$

- Nonlinear dynamicists recognize this immediately as a variant of the Nonlinear Schrödinger equation, here in 3D and with an external potential—more on this later.
- The interaction term—g_0—describes s-wave scattering of atoms
- In 1D case (realizable depending on external potential) we expect (continuum) solitons among excitations.

BECs in OLs

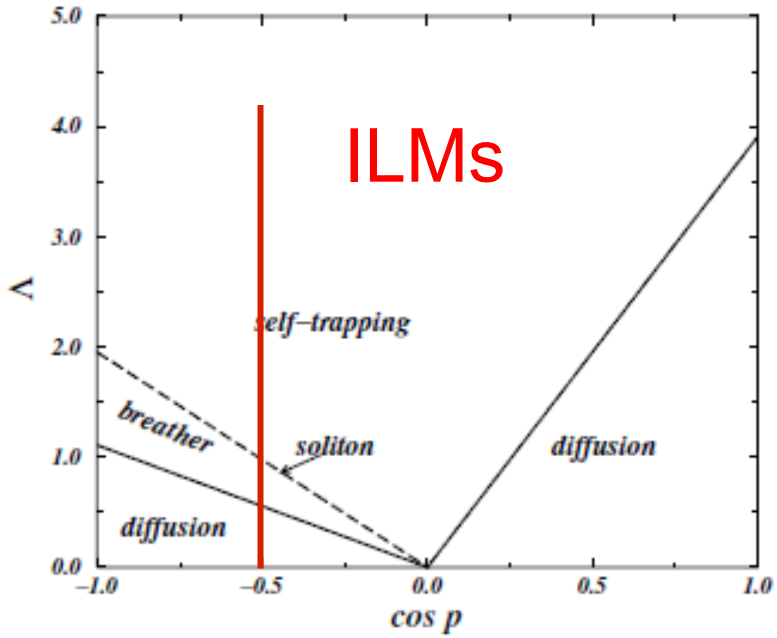
- Quantum system in periodic potential—just as in solid state. Can describe in terms of (extended) Bloch wave functions or (localized) Wannier wave functions, centered on the wells of the potential (but allowing tunneling between the wells.
- Expanding the GPE in terms of Wannier functions leads to the Discrete Nonlinear Schrodinger Equation (DNLSE) for the wave function amplitudes. In normalization we shall later use

$$i\frac{\partial \psi_n}{\partial t} = \lambda|\psi_n|^2\psi_n - \frac{1}{2}[\psi_{n-1} + \psi_{n+1}]$$

- Comes from Hamiltonian (λ=2*U*/*J*)

$$\mathcal{H} = \sum_{n=1}^{M}[U|\psi_n|^4 + \mu_n|\psi_n|^2] - \frac{J}{2}\sum_{n=1}^{M-1}(\psi_n^*\psi_{n+1} + c.c.)$$

Excitations of BECs in OLs



Notice that there exist many kinds of excitations, depending on parameters—*eg*, for *cosp*=-0.5, as Λ ~λN is increased, one sees linear excitations ($\omega \sim k^2$), then localized moving breathers that become solitons along a single line in the plane, then "self-trapped states" = ILMs.

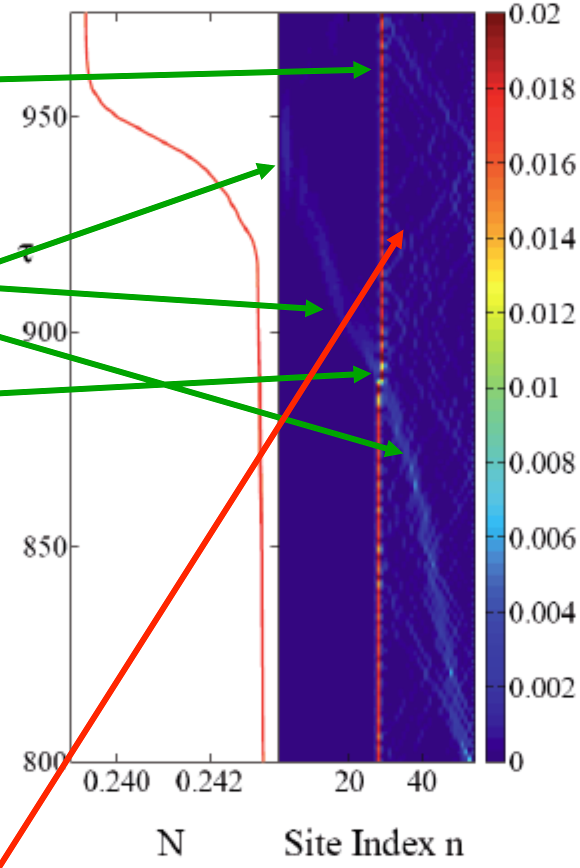
Fig. 2 from A. Trombettoni and A. Smerzi, *PRL* **86**, 2353 (2001)

Intrinsic Localized Modes (ILM)s

- Definition: an “intrinsic localized mode”—is a spatially localized, time-periodic, stable (or at least very long-lived) excitation in a *spatially extended, perfectly periodic, discrete, nonlinear system*.
- Intuition: Discreteness and periodicity imply spectrum of linear oscillations lies in a band of finite width. Nonlinearity implies that the frequency of an oscillation varies with its amplitude (think of pendulum). For a sufficiently large local oscillation, frequency can move out of band and have no possibility of coupling to linear modes=> stable.
- Experiment: ILMs observed in many physical systems.
- Reference: David K. Campbell, Sergej Flach, and Yuri S. Kivshar, “Localizing Energy Through Nonlinearity and Discreteness,” pp. 43-49, *Physics Today* (January 2004).

Excitations of BECs in OLs and their interactions

An event in which a single ILM interacts with a number of moving excitations (right to left) and some amplitude is transmitted through the ILM to the edge of the lattice at N=0. Note that the ILM moves (a single site) in the direction of the incoming excitation. The arrival of the amplitude at n=1 leads to a leaking of the condensate and is called an "avalanche" by the authors. Scattering from ILMs controls the avalanches:
How can we understand this ?



NB: Low energy excitations reflect off ILM

Fig 5 from Ng *et al.*, *New J. Phys* **11**, 073045 (2009).

Summary of Transport through bright ILM

- Results of Ng *et al.* include
  - Little transport through ILM at low energy, excitations reflected
  - Clear threshold energy, E_{thrs}, for significant transport through ILM
  - Above threshold, ILM moves one lattice site in direction of incoming excitation
  - Transport and ILM motion linked to leakage avalanches of BEC from OL
- These results achieved by direct numerical simulation of DNLSE on extended lattices. In remainder of talk, we will give heuristic, mostly analytic explanation of the results.

Heuristic Understanding of ILM Interactions

- ILMs are highly localized and are “self-trapped.” Their properties should follow from small systems.
- Smallest system that allows ILM plus perturbations *and* transport is trimer—already established as useful for studying ILMs.
- DNSLE for trimer (open BCs) becomes

$$\begin{aligned} i\partial_t\psi_1 &= \lambda|\psi_1|^2\psi_1 - \frac{1}{2}\psi_2 \\ i\partial_t\psi_2 &= \lambda|\psi_2|^2\psi_2 - \frac{1}{2}(\psi_1+\psi_3) \\ i\partial_t\psi_3 &= \lambda|\psi_3|^2\psi_3 - \frac{1}{2}\psi_2 \,. \end{aligned}$$

$$N = \sum_{n=1}^{M} |\psi_n|^2 = 1 \,.$$

Analytic approach to DNLSE for Trimer

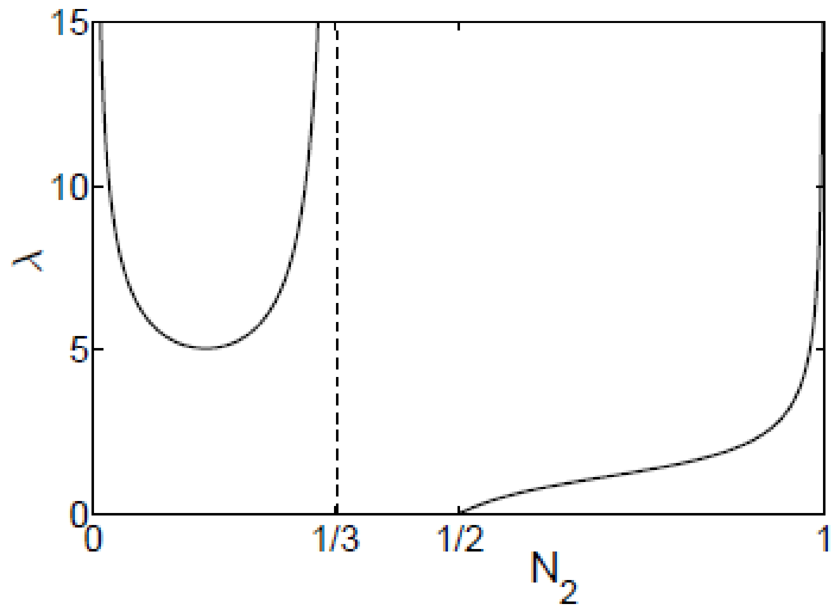
Assuming $\psi_n(t) = A_n e^{iwt}$ leads to

$$\begin{aligned} -wA_1 &= \lambda A_1^3 - \frac{1}{2}A_2 \\ -wA_2 &= \lambda A_2^3 - A_1 \\ 1 &= 2A_1^2 + A_2^2 \,. \end{aligned}$$

$N_i = A_i^2$

$$\lambda(N_2) = \frac{\sqrt{2}(2N_2 - 1)}{\sqrt{N_2(1 - N_2)}(3N_2 - 1)}$$

Three solutions for λ> ~5.04



ILM solutions and heuristic approach

In limit of infinite λ, three types of (symmetrical) solutions are

1) “bright” ILM ,

2) “dark” ILM; and

3) phase-wise and anti-phase wise time-dependent moving solutions

$$\vec{\psi}_{(x3)}(0) = (0,1,0)$$

$$\vec{\psi}_{(x4)}(0) = (\sqrt{1/2},0,\sqrt{1/2})$$

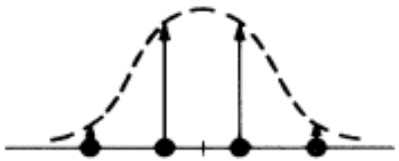
$$\vec{\psi}_{(x5,x6)}(0) = (1/\sqrt{3},\mp 1/\sqrt{3},1/\sqrt{3})$$

Focus on bright ILM solution, perturbed by “excitation” at site 1, study “dynamics” not directly via integration of DNLSE (trivial) but by energy considerations based on “Peierls-Nabarro” barrier and PN energy landscape concepts.

Recall "Peierls-Nabarro" Barrier

Consider a highly localized ILM. As we have seen, it can be centered at a site (a) or between sites (b).

(a)



(b)

A priori, there is no reason that the energies associated with these two positions should be the same. Indeed, Peierls and Nabarro showed that in general they were not, and the "PN barrier" is E_c-E_b, where E_c is the energy of a ILM centered at a site and E_b is energy between two lattice sites. PN barrier may be viewed as minimum barrier to translating the "self-trapped" ILM by a single lattice period.

Peierls-Nabarro Energy Landscape

Definition of PN Energy Landscape: For given A_n , extremize the full Hamiltonian H wrt phase differences: the minimum is the lower part of PN landscape, maximum is the upper part.

$$H = \frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) - (A_1 A_2 \cos(\phi_1 - \phi_2) + A_2 A_3 \cos(\phi_2 - \phi_3))$$

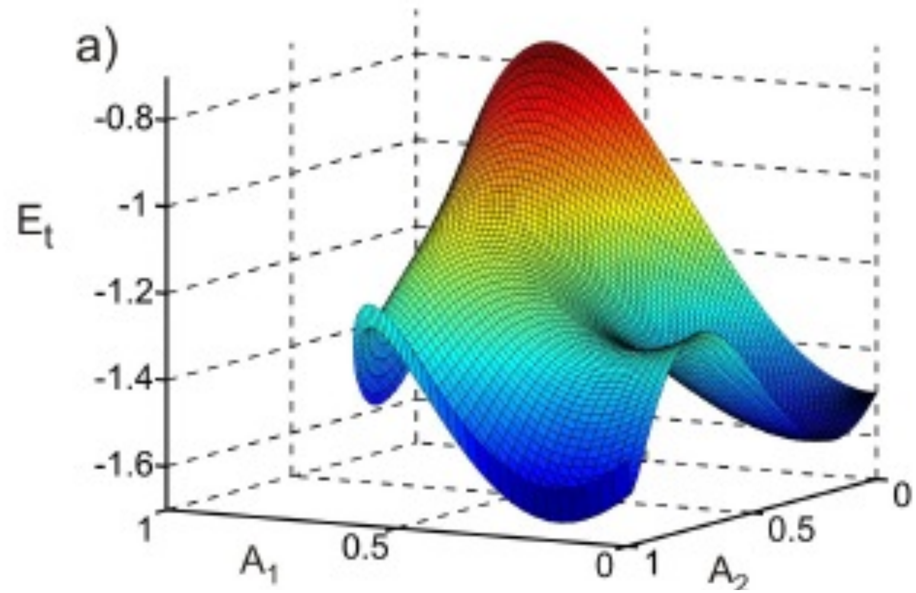
$$H^l_{\mathrm{PN}} = \min_{\delta\phi_{ij}}(-H) \quad ; \quad H^u_{pn} = \max_{\delta\phi_{ij}}(-H)$$

$$H^l_{\mathrm{PN}} = -\frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) - (A_1 + A_3)A_2$$

$$H^u_{\mathrm{PN}} = -\frac{\lambda}{2}(A_1^4 + A_2^4 + A_3^4) + (A_1 + A_3)A_2 \ .$$

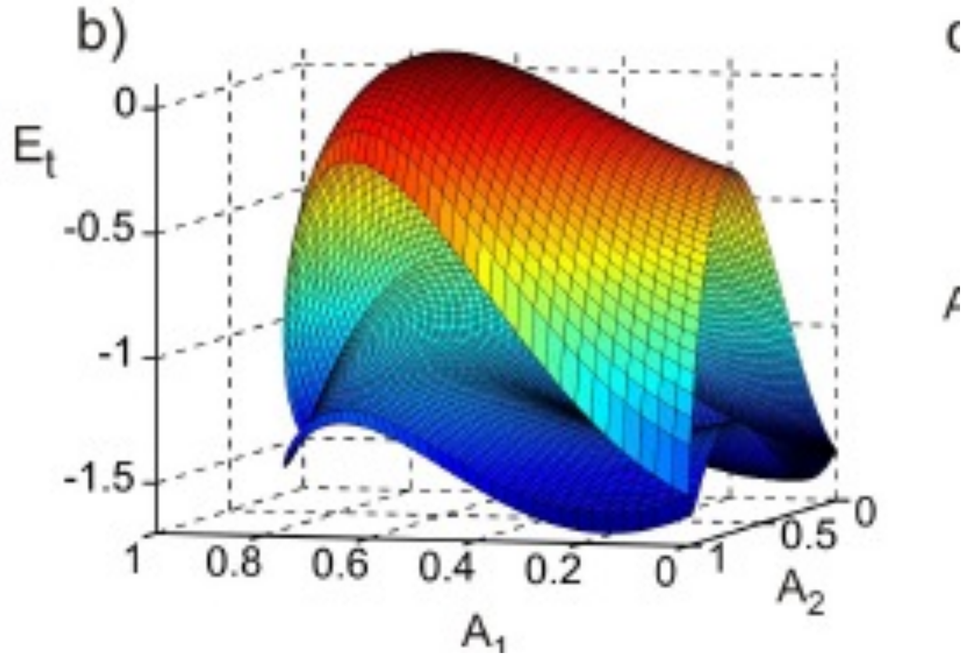
Images of PN Energy Landscapes (next slides)

Peierls-Nabarro Energy Landscape



The lower “sheet” of the PN Energy Landscape for $\lambda = 3$; there are three minima separated by saddle points.

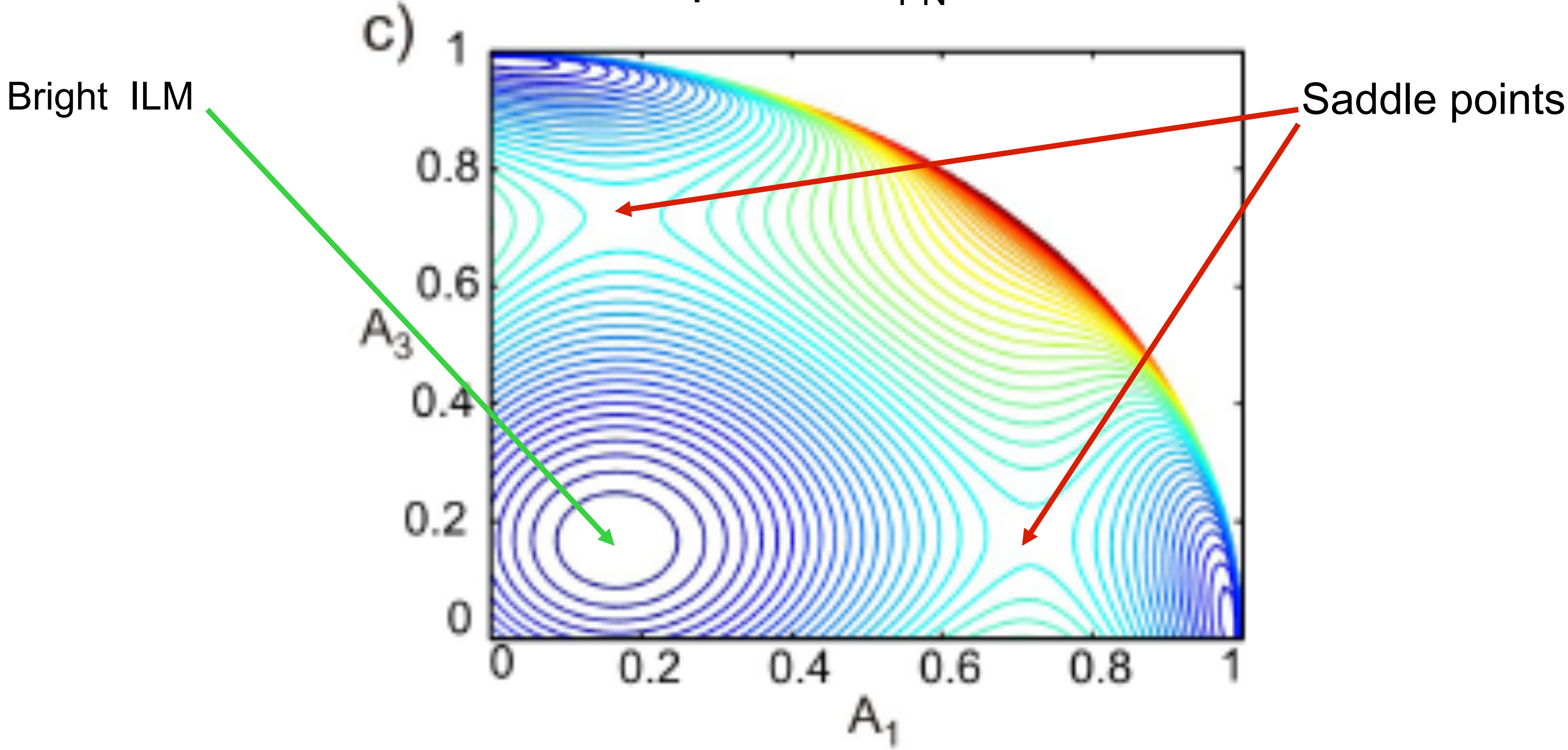
Peierls-Nabarro Energy Landscape



The phase space of the trimer is restricted by the PN “shell” which exists between the lower and upper sheets for PN Energy Landscape (figure for λ=3).

Peierls-Nabarro Energy Landscape

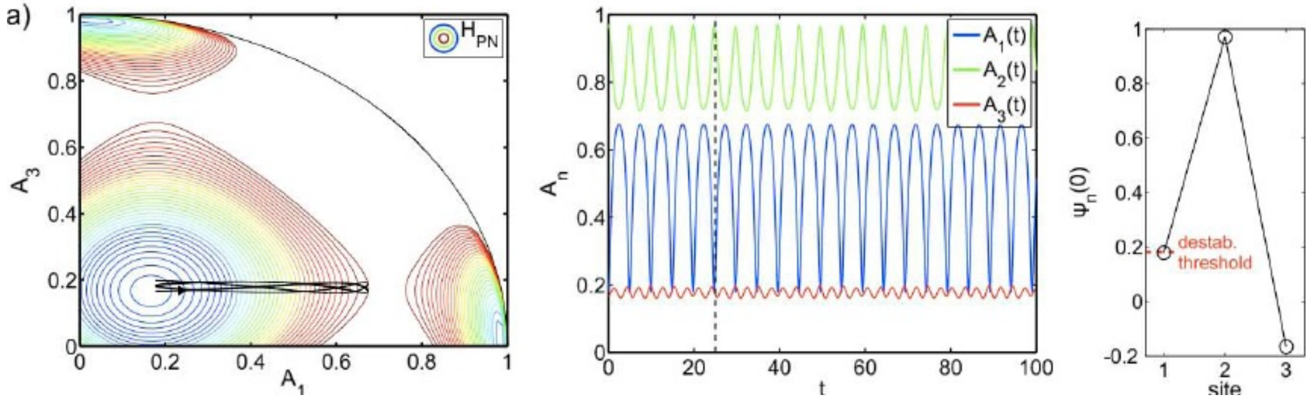
Contour plot of H^{I}_{PN} for $\lambda=3$

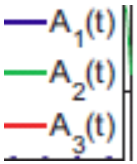


Study dynamics after initial perturbation of form

$$\vec{\psi}(0) = (-(A_1^b + \delta_A)e^{i\delta_\phi}, A_2, -A_1^b),$$

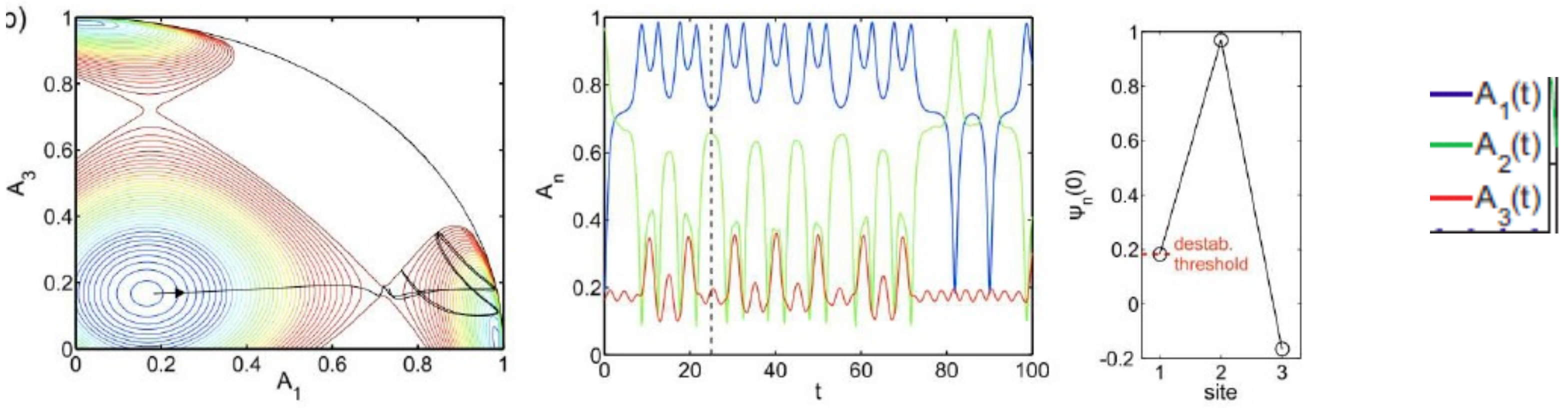
Dynamics on the PN Landscape





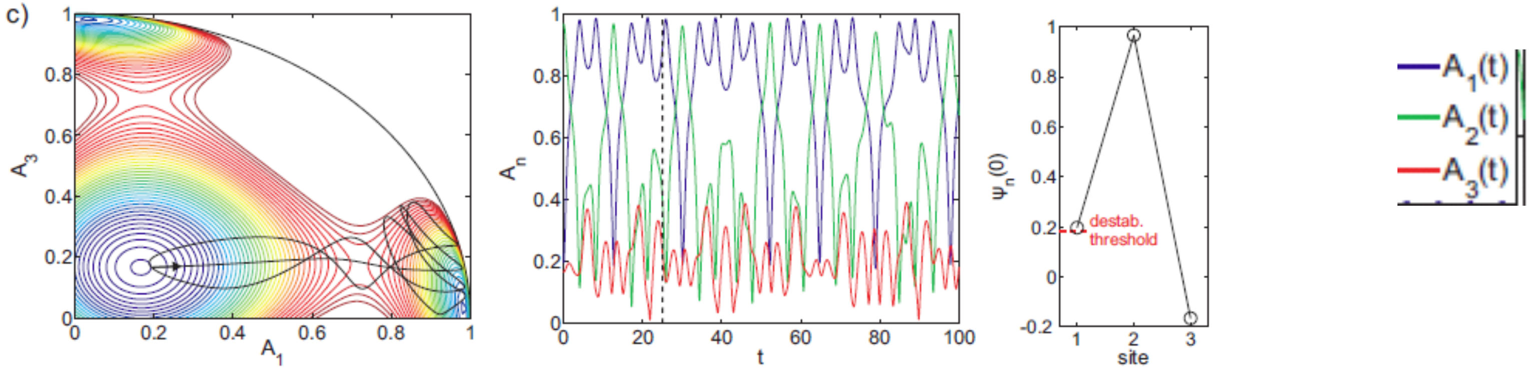
The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for . Note that the total energy E_t is below the threshold ~ height of saddle point so that the motion is confined to a “region” around the bright breather. The solution is plotted for 25 units (see vertical dashed line). The right panel shows that the energy of the initial condition (E_t=-1.32) is below E_{thrs} (-1.31).

Dynamics on the PN Landscape



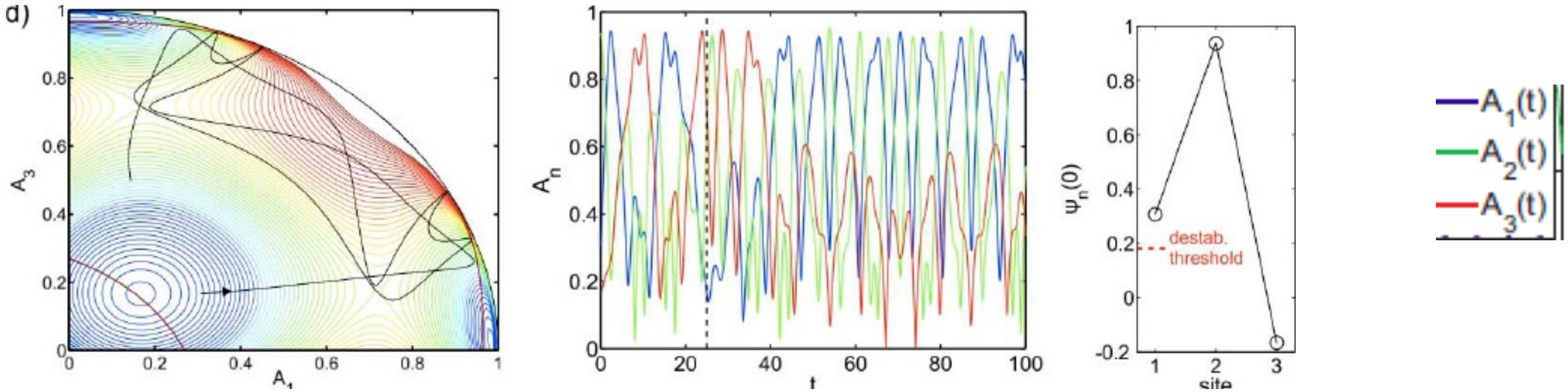
The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for E_t=E_{thrs}=-1.31; we see that the rim of the PN landscape clearly restricts the dynamics and governs the destabilization of the ILM. The size of right "lobe" gives the maximum of A_3, the amplitude transferred to site 3.

Dynamics on the PN Landscape



The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for E_t=-1.28. Note that the bottleneck at the rim widens and the amplitude transferred to site 3 is increased.

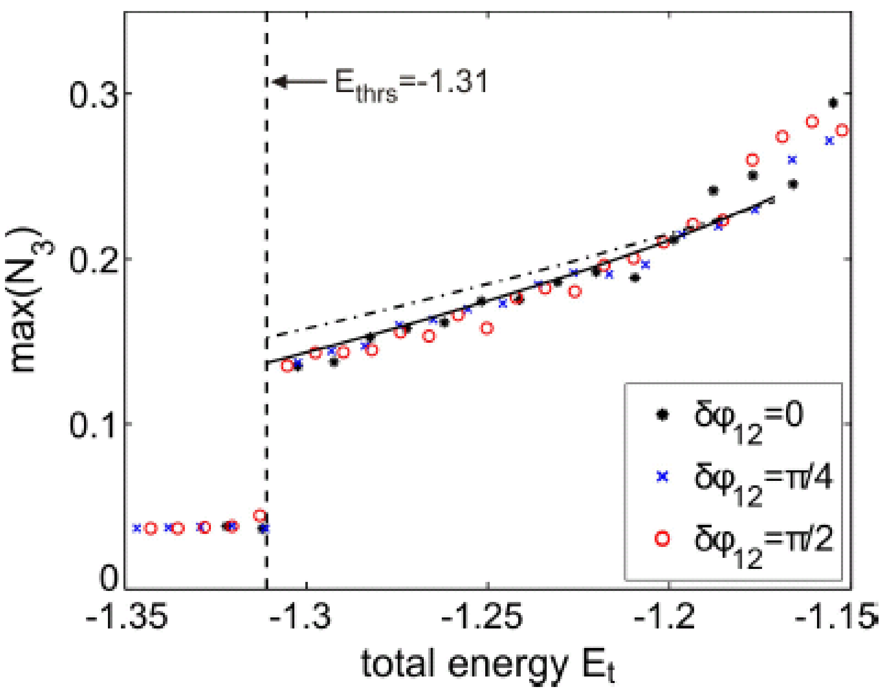
Dynamics on the PN Landscape



The exact motion (black lines) of the trimer DNLSE plotted on the contour plot of the PN energy landscape in the A_1-A_3 plane as a function of time for E_t=-1.04, well above E_{thrs}. The motion explores large parts of the phase space and visits all three sites. The red line indicates where the upper PN energy landscape limits the motion; note that it prohibits A_n values close to 1.

Norm Transfer vs. Total Energy

Maximum value of the norm at site 3 after a collision of a ILM at site 2 with a lattice excitation “coming” from site 1 as a function of total energy on the trimer. For $E_t<E_{thrs}$ ~ -1.31 (for λ=3) the ILM is stable and almost no transfer occurs. For $E>E_{thrs}$, ILM is unstable, moves to site 1, and the transmitted amplitude rises sharply (and is independent of the phase difference).



Solid line=numerics. Dashed line = analytic result from PN energy landscape calculation.

Summary and Conclusions

- The threshold for collisions leading to “tunneling” of amplitude through ILM is sharp and is related to the loss of stability of the ILM.
- The loss of stability of the ILM is related to the PN barrier in which the ILM is self-trapped.
- When amplitude is transmitted, the ILM moves by one site in the direction of the incoming wave.
- The dynamics of the interactions of moving excitations with ILMs, including the size of the transmitted amplitude and the motion of the ILM, can be understood in terms of motion on the fixed energy contours of a PN Energy Landscape.
- The nonlinear trimer provides a quantitative fit to the results observed in extended systems and allows us to obtain an analytic expression for the transmitted amplitude.

Selected References

S. Aubry, "Discrete Breathers: Localization and transfer of energy in discrete Hamiltonian nonlinear systems," *Physica D* **216**, 1-30 (2006).

David K. Campbell, Sergej Flach, and Yuri S. Kivshar, "Localizing Energy Through Nonlinearity and Discreteness," pp. 43-49, *Physics Today* (January 2004).

J. Esteve et al, "Squeezing and entanglement in a Bose-Einstein condensate," *Nature* **455**, 1216-1219 (2008),

H. Hennig, J. Dorignac, and D. K. Campbell "Transfer of BECs through discrete breathers in optical lattices," *Phys. Rev. A* ***82*** 053604 (2010) .

Y.Kivshar and D. K. Campbell, "Peierls-Nabarro potential barrier for highly localized nonlinear modes," *Phys. Rev. E* **49**, 3077 (1993).

Yuri Kivshar and Sergej Flach, guest editors, *Chaos* **13**, #2 Focus Issue: "Nonlinear Localized Modes: Physics and Applications,"

R. Livi, R. Franzosi, and G-L. Oppo, "Self-localization of Bose-Einstein Condensates in Optical Lattiices via Boundary Dissipation," *Phys. Rev. Letters* **97**, 060401 (2006).

G.S. Ng et al, "Avalanches of Bose-Einstein condensates in leaking optical lattices," *New J. Phys.* **11**, 073045 (2009).

A. Trombbettoni and A. Smerzi "Discrete Solitons and Breathers with Dilute Bose-Einstein Condensates," *Phys. Rev. Lett.* **86**, 2353 (2001)

QUESTIONS ?

BEC in an Optical Lattice

Counter-propagating laser pulses creating standing wave that interacts with atoms, so that the BEC experiences a periodic potential of the form:

$$V_{\text{ext}}(\vec{r}) = U_L(x, y) \sin^2[2\pi z / \lambda]$$

$U_L(x,y)$ is transverse confining potential, λ is the laser wavelength (typically 850 nm) and "*z*" is the direction of motion.

Image from I. Bloch *Nature* **453** 1016-1022 (2008)

BECs in Optical Lattices

- Quantum system in periodic potential—just as in solid state. Can describe in terms of (extended) Bloch wave functions or (localized) Wannier wave functions, centered on the wells of the potential (but allowing tunneling between the wells.
- Expanding the GPE in terms of Wannier functions leads to the Discrete Nonlinear Schrodinger Equation (DNLSE) for the wave function amplitudes. In normalization we shall later use

$$i\frac{\partial \psi_n}{\partial t} = \lambda|\psi_n|^2\psi_n - \frac{1}{2}[\psi_{n-1} + \psi_{n+1}]$$

- Comes from Hamiltonian ($\lambda=2U/J$)

$$\mathcal{H} = \sum_{n=1}^{M}[U|\psi_n|^4 + \mu_n|\psi_n|^2] - \frac{J}{2}\sum_{n=1}^{M-1}(\psi_n^*\psi_{n+1} + c.c.)$$

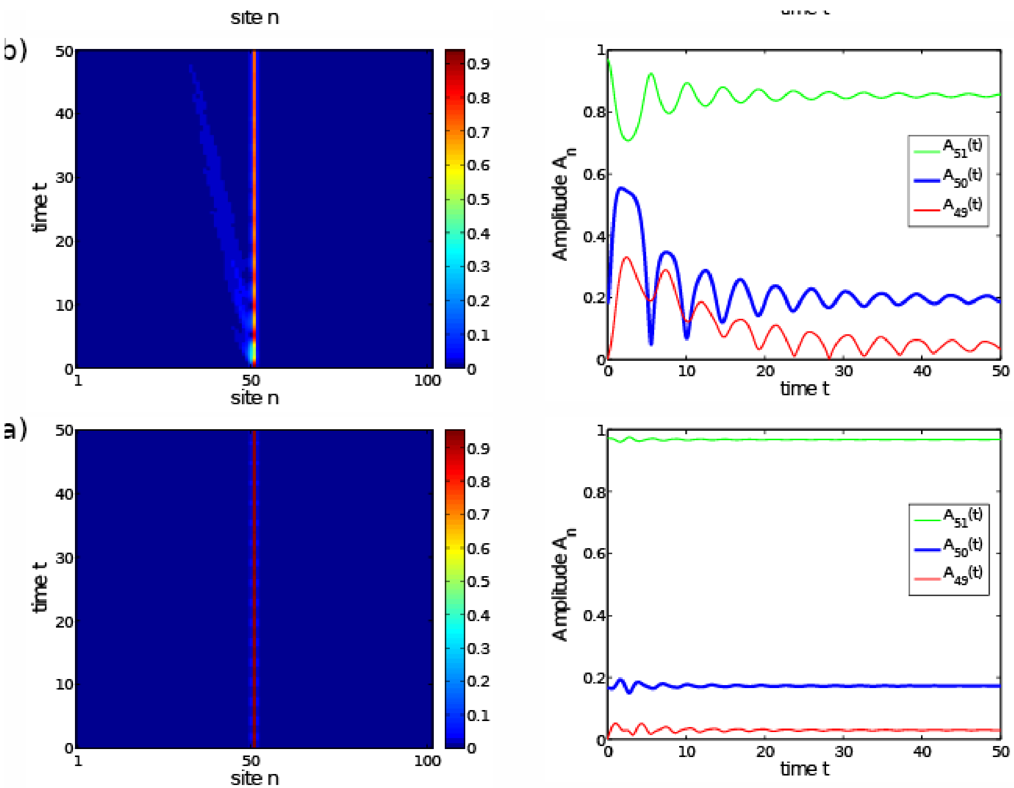
Interlude on “Intrinsic Localized Modes”

- Definition: an “intrinsic localized mode”—is a spatially localized, time-periodic, stable (or at least very long-lived) excitation in a *spatially extended, perfectly periodic, discrete system*.
- Bottom Line: The mechanism that permits the existence of ILMs has been understood theoretically for nearly two decades. Only in last decade have they been observed in physical systems as distinct as charge-transfer solids, Josephson junctions, photonic structures, micromechanical oscillator arrays, and BECs trapped in optical lattices.

Results on Extended Optical Lattices

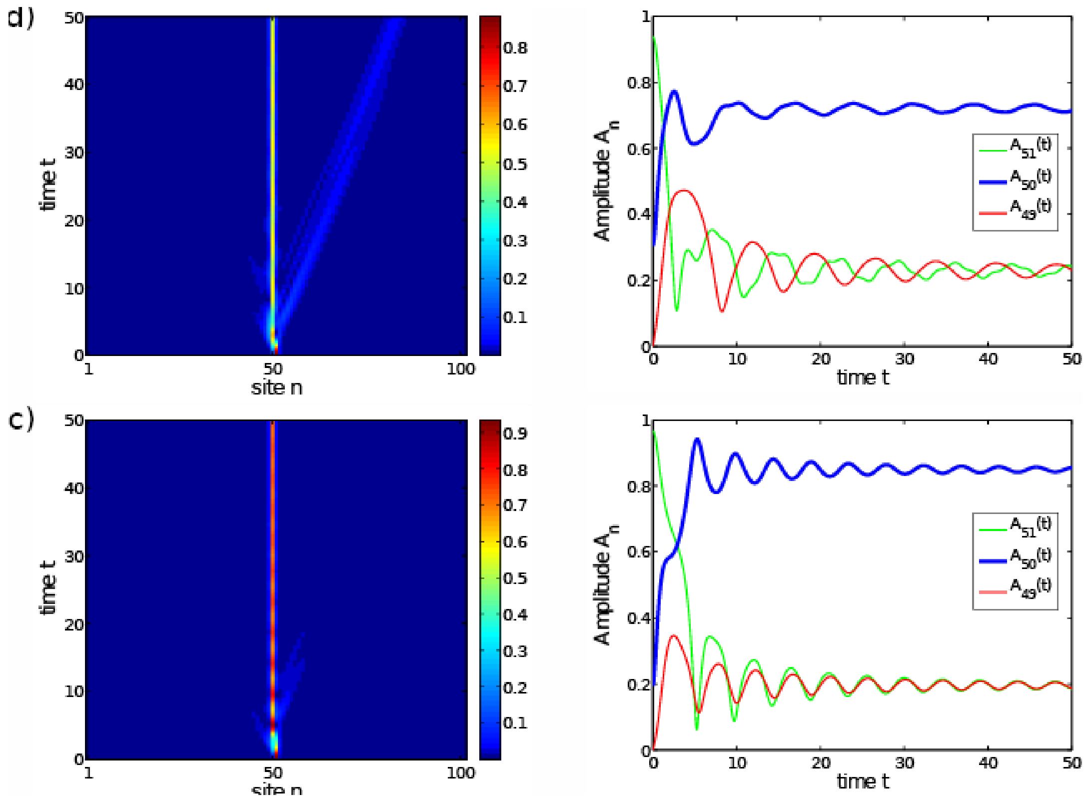
To test mechanism on extended lattices, we study the case N=101 with a ILM of the form studied in the nonlinear trimer located at the center of the lattice (N_0 =51). In the next two slides, we show the results of increasing the total energy on the trimer. We find that in the extended system, the energy threshold for destabilization and consequent motion of the ILM is slightly *higher* than in the trimer—we attribute this to the existence of many other degrees of freedom in the extended system. However, at energies very close to the threshold predicted from the trimer (E_{thrs}), the destabilization and migration of the ILM do occur, confirming that the mechanism that we have described analytically in the trimer does capture the behavior of the extended system.

Results at low energy



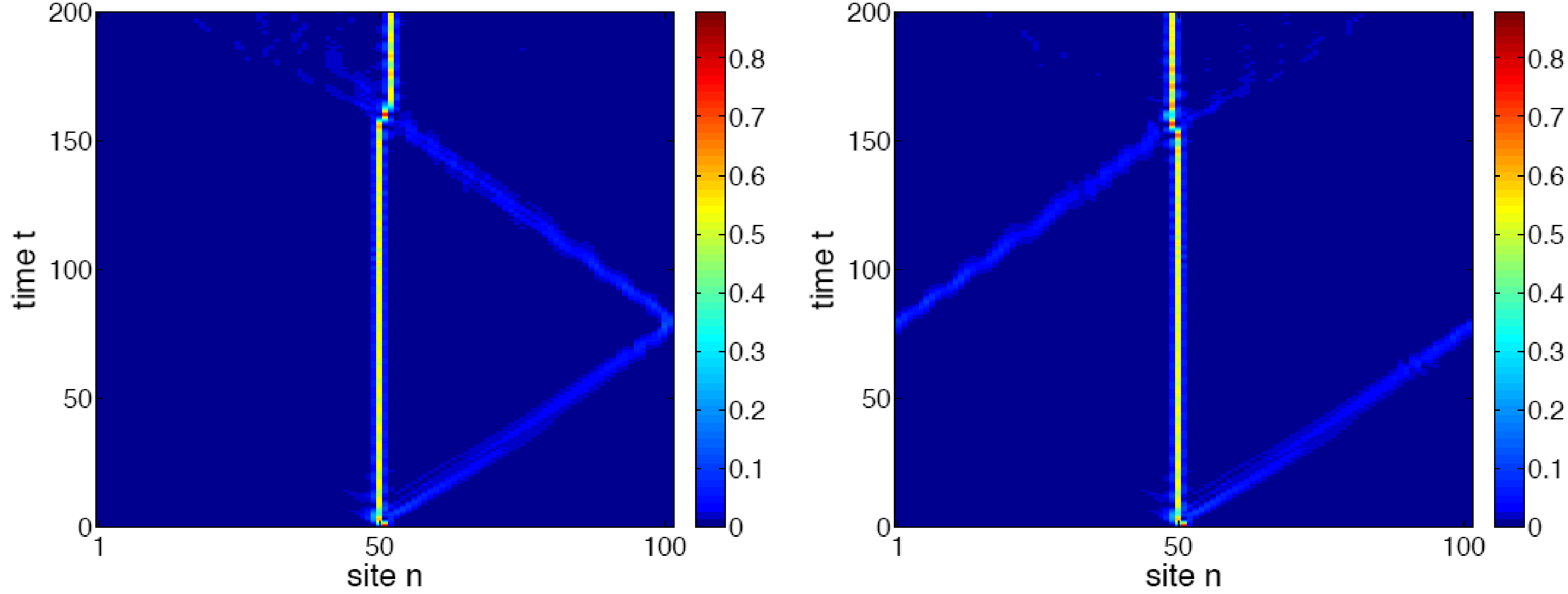
- N=101 lattice. a) unperturbed “trimer” ILM at site 51; b) Perturbation on site 50 is just above trimer threshold. E_{thrs}; the ILM on extended lattice remains (barely) stable and the energy is reflected off to left as a low amplitude, seemingly localized excitation. The right panels confirm that in neither case does the ILM move.

Results at higher energy



- N=101 lattice. c) ILM migrates by one site towards the perturbation after t ~ 5 time steps; d) ILM migrates at t~ 2 time steps and a localized excitation (moving breather) moves off to right, indicating the passage of energy through the ILM. Note that panels on right confirm that ILM has moved.

Results for Longer Times



- N=101 lattice. This shows results for longer time. Left panel shows reflection of moving breather off the open boundary and then second interaction with ILM leading to two site migration. Right panel shows periodic boundary conditions (so discrete pseudomomentum is conserved.) Note that in each case, ILM moves in direction of incoming wave, as predicted by PN analysis of trimer.

Los Alamos—1983

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→DC

0990

Zürich, 20 November 1982

Dear Alan,

Thank you for your letter and the kind offer to arrange a visit in Los Alamos. In the mean-time, unfortunately, I had to realize that it will probably not be so easy to come for several days, this time, for the following reasons. Maki has offered me to stay two months at the USC. On the other hand I have been engaged to give a series of lectures at the University of Pavia, very likely from the middle of February on. Therefore I think that I should predominantly stay at the USC during the six weeks of my visit. If possible (and not too expensive) I may come for a long week-end; therefore I enclose the information you required.
The work on the impurity problem is proceeding quite well; it turns out that even the simple limiting case where lattice relaxation is neglected has some unexpected features, which are related to the phase coherence of the unperturbed electron wavefunctions. This brought me back to the phase problem you raised during my last visit. A complex order parameter is obtained by allowing the Fermi level to become space-dependent, formally by the following replacements,

$$\psi_1(x) \to e^{i\varphi(x)}\psi_1(x)\,, \quad \psi_2(x) \to e^{-i\varphi(x)}\psi_2(x)\,.$$

I hope still to seeing you soon and discussing with you about all that (including the strange busy life of a free-land physicist).

With best regards

Dionys

Dionys Baeriswyl

Intrinsic Localized Modes (ILM)s

- Definition: an "intrinsic localized mode"—is a spatially localized, time-periodic, stable (or at least very long-lived) excitation in a *spatially extended, perfectly periodic, discrete, nonlinear system*.
- Intuition: Discreteness and periodicity imply spectrum of linear oscillations lies in a band of finite width. Nonlinearity implies that the frequency of an oscillation varies with its amplitude (think of pendulum). For a sufficiently large local oscillation, frequency can move out of band and have no possibility of coupling to linear modes=> stable.
- Experiment: ILMs observed in many physical systems.

Another Image of BEC

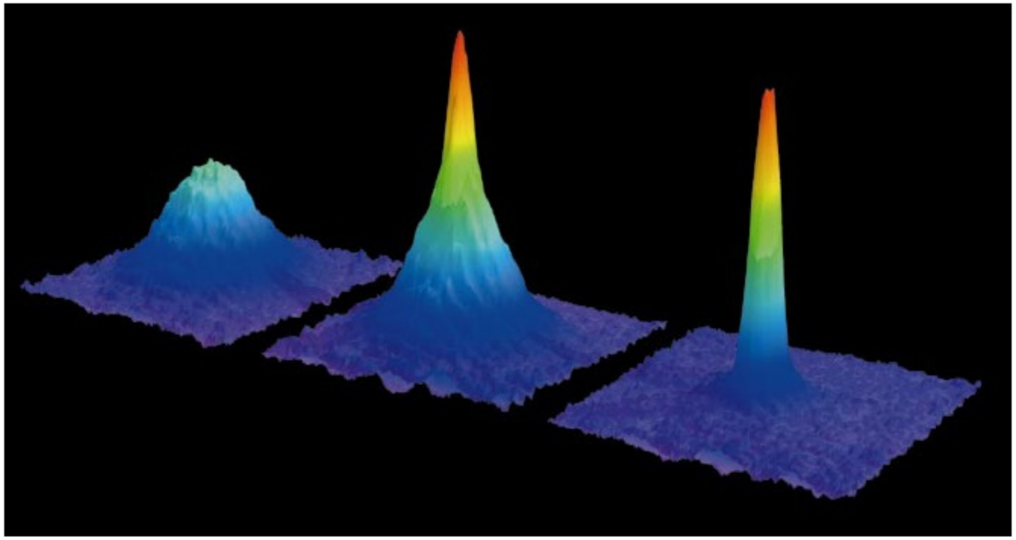


FIG. 7. Observation of Bose-Einstein condensation by absorption imaging. Shown is absorption vs two spatial dimensions. The Bose-Einstein condensate is characterized by its slow expansion observed after 6 ms time of flight. The left picture shows an expanding cloud cooled to just above the transition point; middle: just after the condensate appeared; right: after further evaporative cooling has left an almost pure condensate. The total number of atoms at the phase transition is about 7×10^5, the temperature at the transition point is 2 μK [Color].

W. Ketterle, Nobel Lecture *RMP* **74** p 1140 (2002)

ILMs: Intuition and Theory

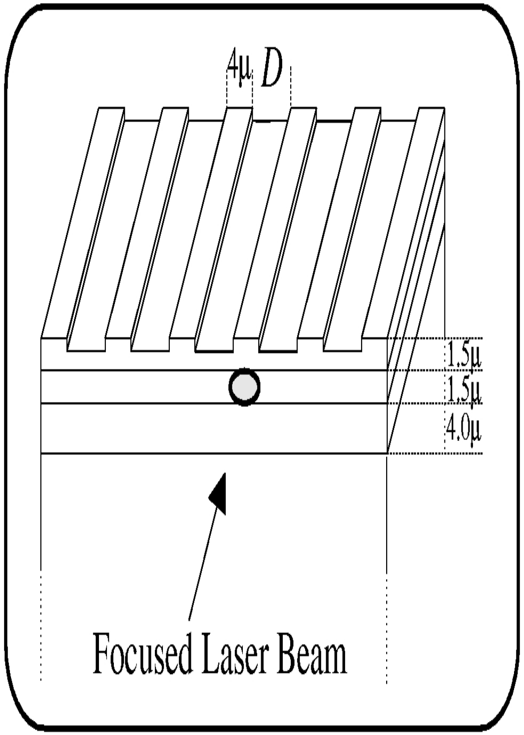
- Consider diatomic molecule modeled as two coupled *anharmonic* oscillators:
  - *anharmonic* = nonlinear => frequency depends on amplitude of motion, $\omega(A)$: familiar from plane pendulum, where frequency *decreases* with amplitude
- Consider limit of no coupling:
  - Trivial to “localize” excitation on one of oscillators only; frequency of oscillation depends on amplitude ~ energy

ILMs: Intuition and Theory

- Consider weak coupling:
  – Imagine one oscillator highly excited, other weakly excited. Frequencies are very different. Suppose $\omega(A_1) / \omega(A_2) \neq p/q$—i.e., frequencies are incommensurate. Then there are no possible resonances between oscillators and energy transfer must be very difficult, if even possible.
- Can formalize this heuristic argument via KAM Theorem

Experiments in Optical Waveguides

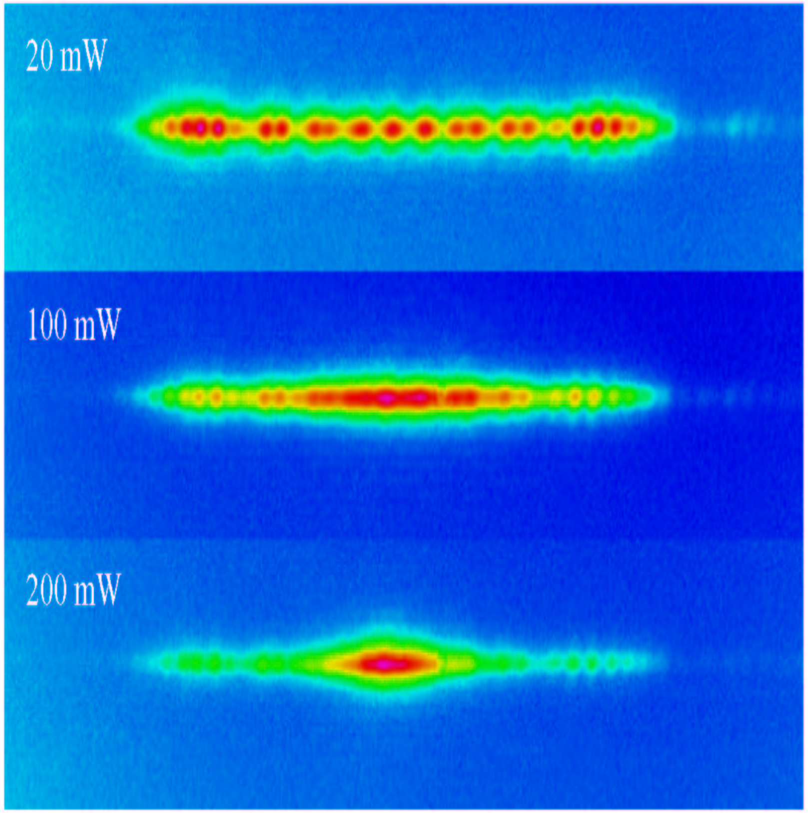
A schematic view of an optical waveguide array created by patterning a layered semiconductor, showing the rough dimensions of the system. Note that the input laser beam can be focused on a single element of the array, corresponding to an initially spatially localized excitation, which then propagates toward the output facet at the back of the array. (Eisenberg *et al. Phys. Rev. Lett*. **81**, 3383-3386 (1998)).



Experiments in Optical Waveguides

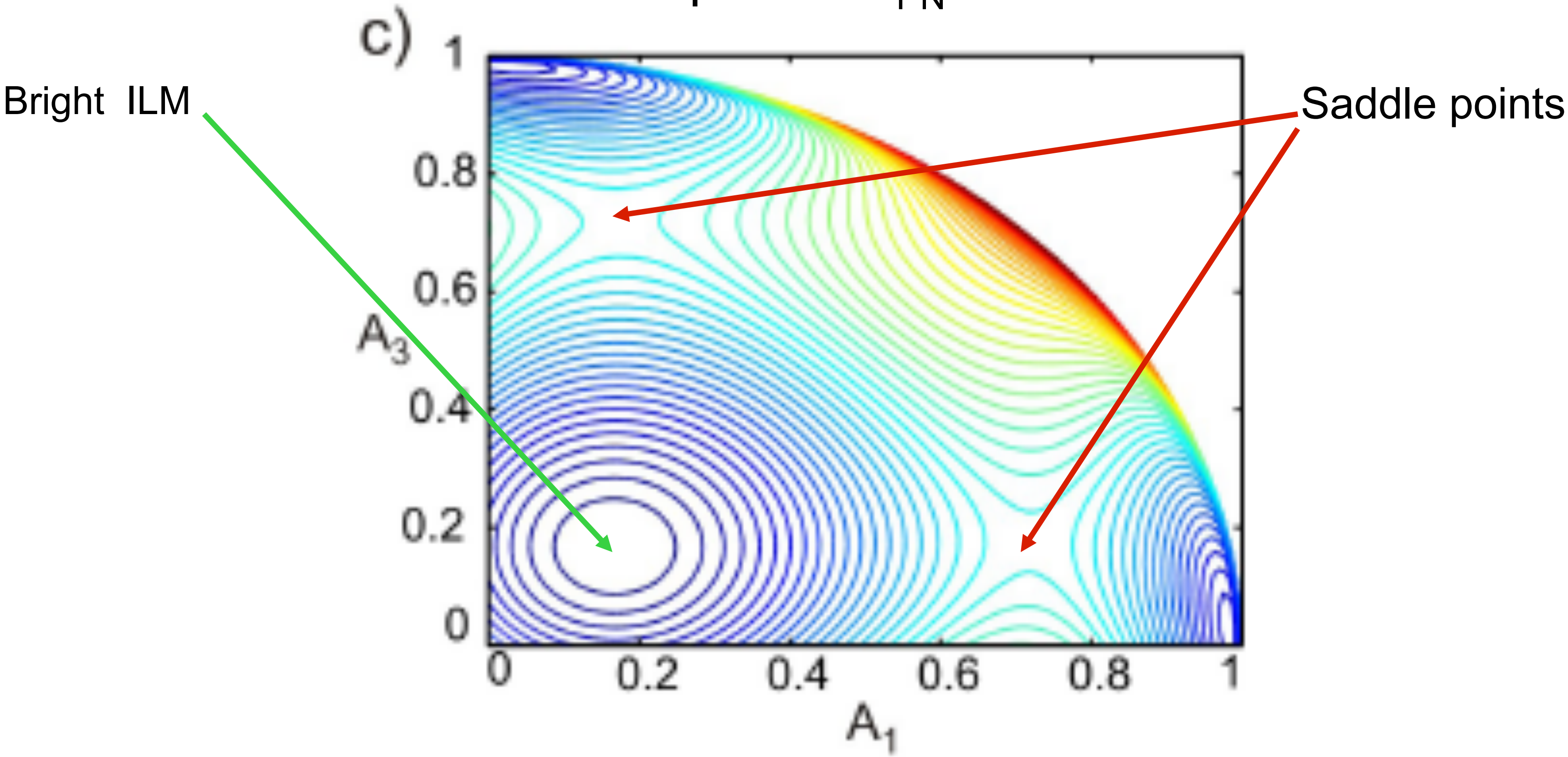
Edge-on view of the output facet of the coupled optical waveguide array shown on previous slide. The input pulse is localized at the center of the array. At low power, pulses propagate linearly and "diffract" across entire array. At intermediate power, nonlinear effects induce some localization. At high power, the pulse remains truly localized and is an example of am ILM in these systems.

(After Eisenberg *et al. Phys. Rev. Lett.* **81**, 3383-3386 (1998)).



Peierls-Nabarro Energy Landscape

Contour plot of H^{I}_{PN} for $\lambda=3$



Next 4 slides show dynamics after initial perturbation of form

$$\vec{\psi}(0) = (-(A_1^b + \delta_A)e^{i\delta_\phi}, A_2, -A_1^b),$$