Rare fluctuations and large-scale circulation cessations in turbulent convection

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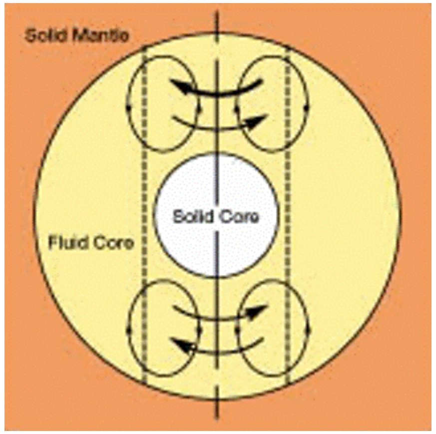
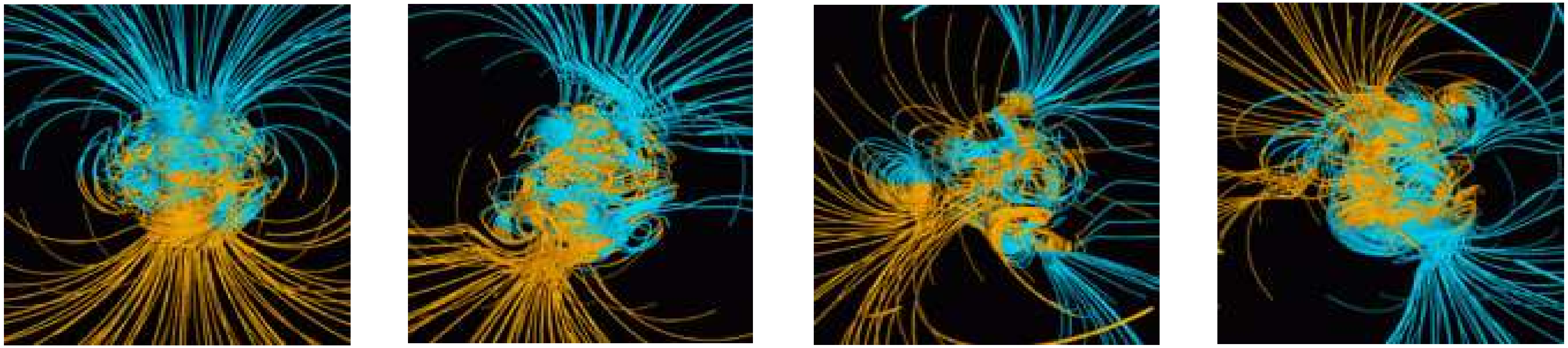
- Earth magnetic field is due to liquid iron flow in outer core

- Sustaining magnetic field requires constant energy pump: **large scale circulation** (LSC) brings warmer fluid from inner core upwards bringing colder fluid downwards



- Rare events of **cessation** of this LSC may lead to reversal of Earth's magnetic field observed ~ every 150,000 years



Glatzmaier and Roberts, Nature (1995)

- Cessations or even weakening of the ocean's thermohaline circulation may cause considerable temperature variations leading to catastrophic events

Marshall and Schott, Rev. Geophys. (1999)

Is there a model system exhibiting these phenomena?

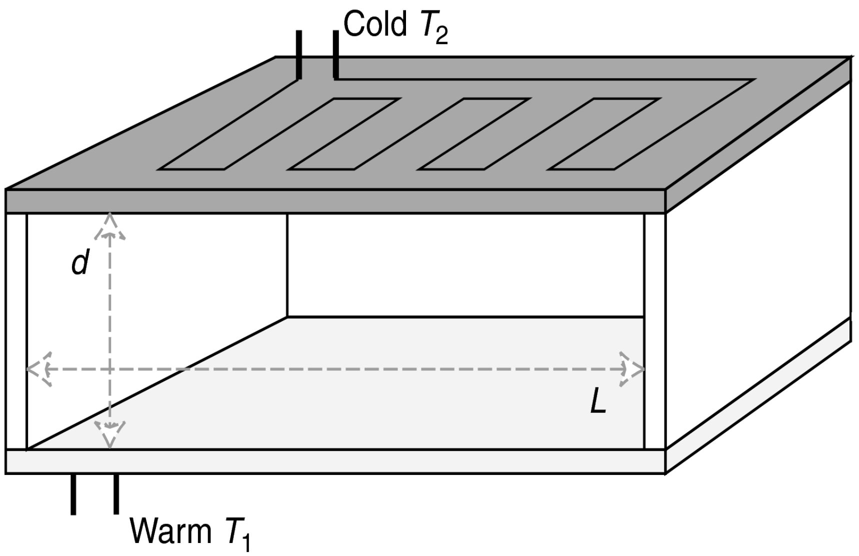
Yes. Within realm of Rayleigh-Bénard convection

RBC displays large-scale circulation and cessations and can be experimentally studied in a controlled manner

Outline

- Introduction to Rayleigh-Bénard convection (RBC)
- Governing hydrodynamic equations of RBC
- Large scale circulation (LSC) dynamics: two coupled stochastic differential equations
- Analytical solutions and comparison with experimental data

RBC - Consider a container heated from below



a well-controlled experimental system that can mimic natural LSCs

Lord Rayleigh (1842-1919)

Henry Bénard (1874-1939)

$$Ra \equiv \frac{\text{buoyancy force}}{\text{diffusive forces}} = \frac{\alpha g \Delta T d^3}{\nu \kappa}$$

controls flow nature

α - thermal expansion coefficient; ν - kinematic diffusivity

κ - thermal diffusivity; g - gravitational acceleration

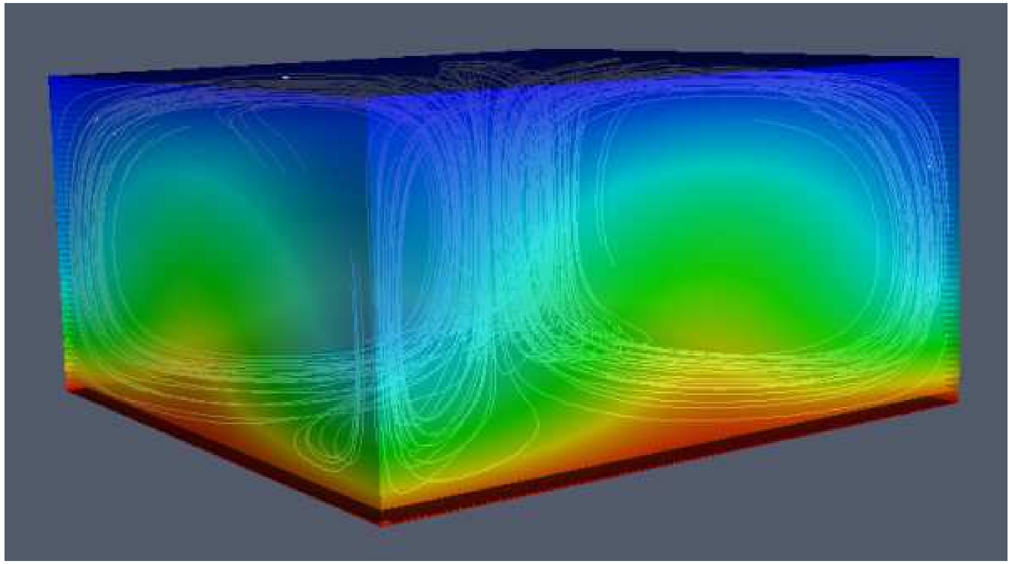
Viscosity prevents motion. Thermal diffusion reduces temp gradient that drives convective motion via buoyancy

For $Ra < Ra_c$: no-flow conduction state

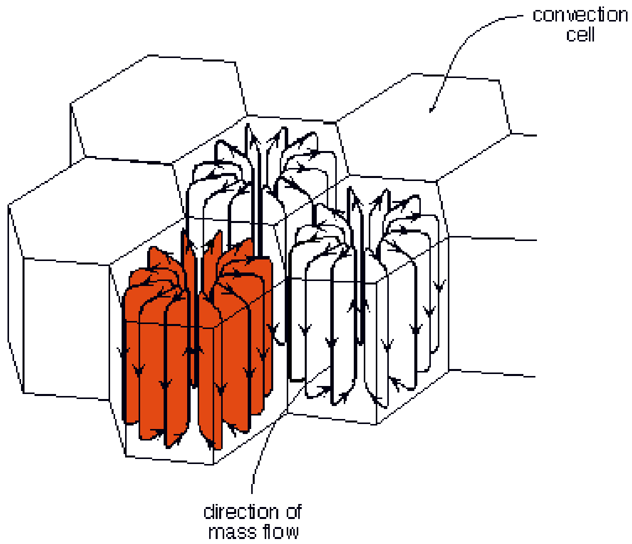
For $Ra > Ra_c$: convection-dominated flow

linear stability analysis yields $Ra_c \approx 1700$

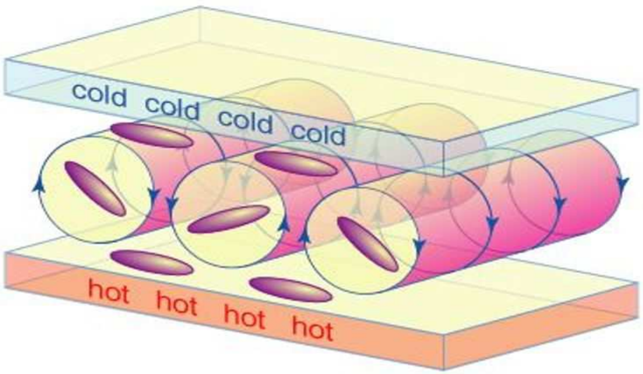
Rayleigh-Bénard convection

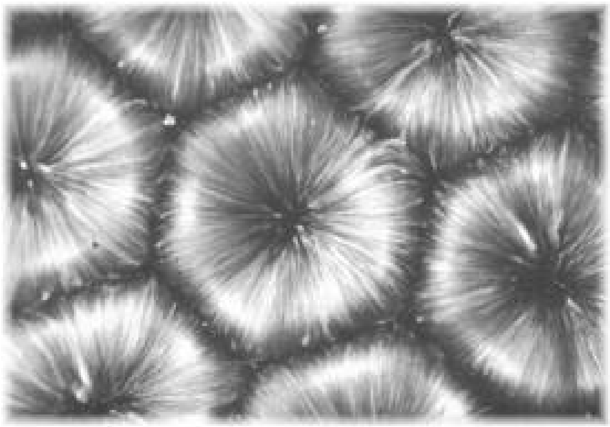


3-D simulation of RB convection for Ra=10^4, Pr=1



$$1700 < Ra < 10^5$$





Van Dyke (1982)

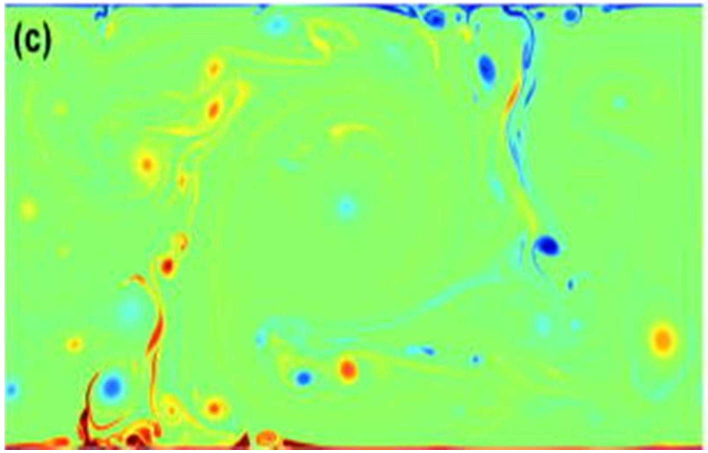
Flow via convection cells moving hot fluid upwards and cold fluid downwards

$Ra > 10^5$ Turbulent flow: streamlines disappear, fields intermittent

$Ra > 10^6$ **Coherent LSC ("wind of turbulence")**: carries warm fluid from the bottom plate up one side of the sample which then cools when passes the top plate and goes down on opposite side of the sample

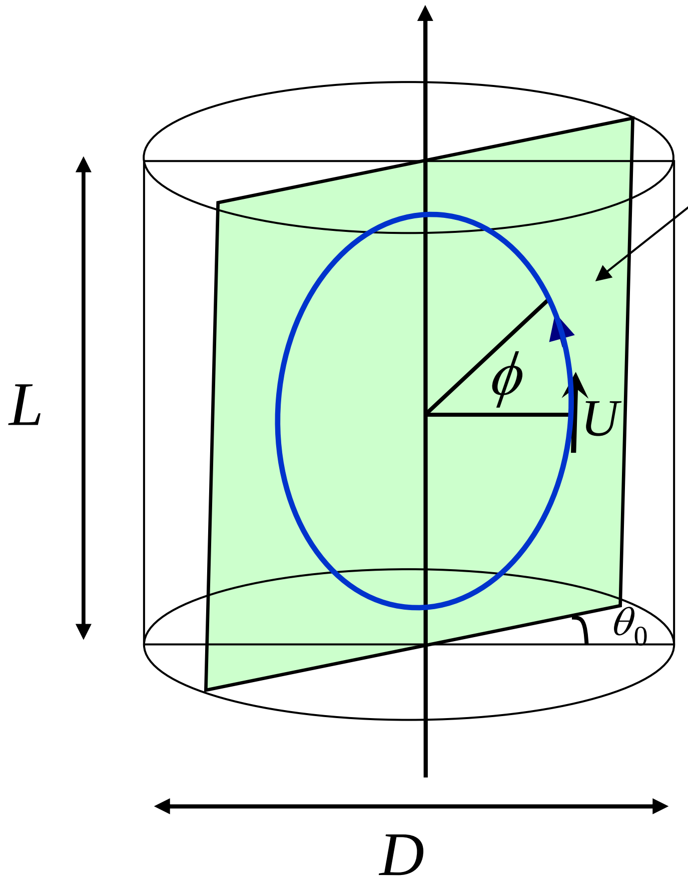
Krishnamurti and Howard, PNAS (1981), Kadanoff, PT (2001)

Xi, Lam and Xia, JFM (2004)



Johnston and Doering, Chaos (2007)

Experiment with water, $Ra=3.7\cdot10^8$: Du and Tong, JFM (2000)



LSC plane

$$\frac{\partial u_\phi}{\partial t}+(\mathbf{u}\cdot\nabla)u_\phi \approx \nu\nabla^2 u_\phi + \alpha g(T-T_0)\cos\phi$$

azimuthal plane

$$\frac{\partial u_\theta}{\partial t}+(\mathbf{u}\cdot\nabla)u_\theta \approx \nu\nabla^2 u_\theta$$

LSC has stochastic nature:

- random cessations
- random diffusion and abrupt reorientations of LSC plane following cessation

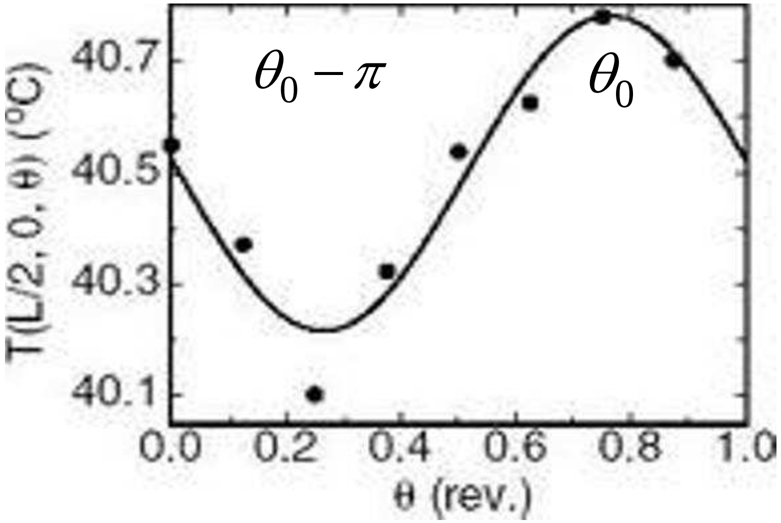
Instead of the velocity, **azimuthal temperature** is a measure of LSC

At $\theta = \theta_0$ hot plumes rise; at $\theta = \pi + \theta_0$ cold plumes sink

$$T = T_0 + \delta \cos(\theta_0 - \theta)$$

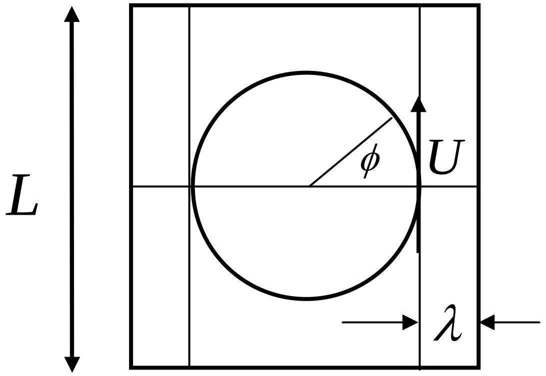
δ measures azimuthal deviations from T_0

effectively measures LSC amplitude



Brown and Ahlers, JFM (2006, 2008)

It has been shown that $U \sim \delta$

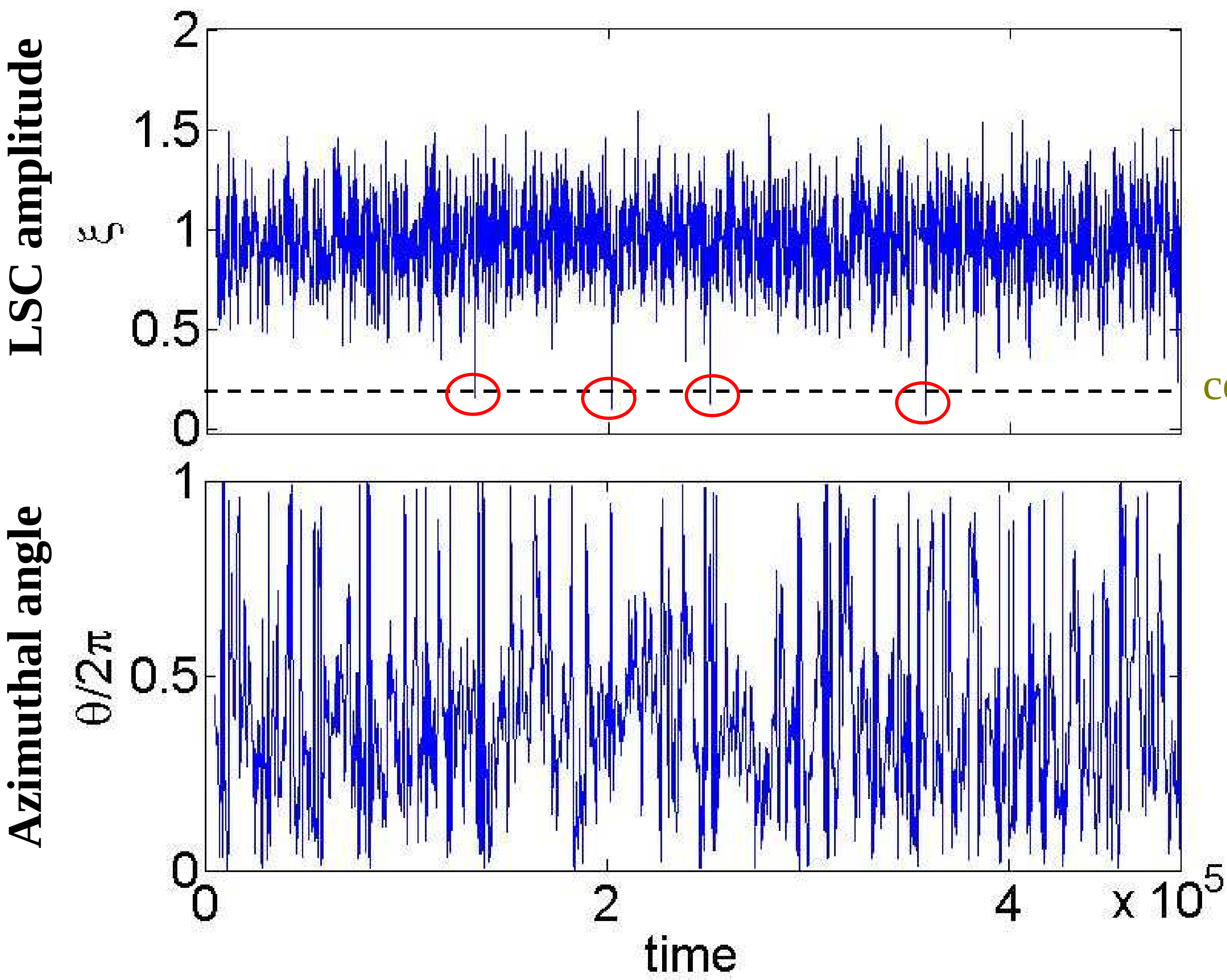


Brown and Ahlers, JFM (2008)

"Hand waving argument": LSC is buoyancy-driven

$$\dot{U} \sim \frac{U}{\tau_\delta} \sim \alpha g \Delta T \sim \alpha g \delta \quad \Rightarrow \quad U \sim \delta$$

τ_δ LSC turnover time; $\mathrm{Re} = L^2 / \nu \tau_\delta$



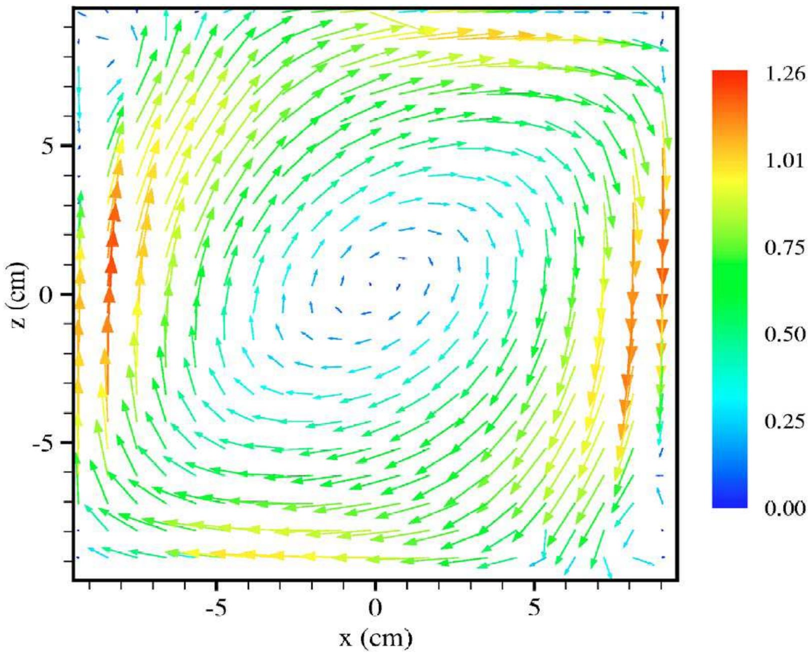
cessation threshold

Redrawn from experimental data Brown and Ahlers (2008)

$$Ra = 1.1 \cdot 10^{10}$$

$$\mathrm{Pr} = 4.38$$

Cessations are rare events. However due to their importance, we want to accurately estimate how rare is rare!

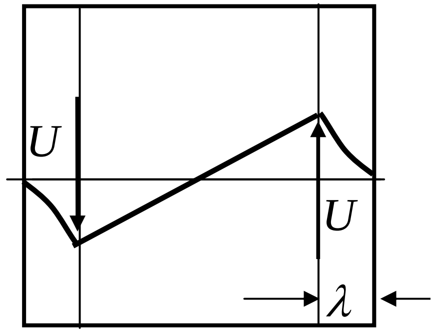


Sun, Xia and Tong, PRE (2005)

Navier-Stokes equation for u_ϕ :

$$\frac{\partial u_\phi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\phi = \nu\nabla^2 u_\phi + \alpha g\left(T - T_0\right)\cos\phi$$

In the bulk $\quad u_\phi = \dfrac{2rU}{L}$



Coarse grained description

$$\dot{u}_\phi \sim \dot{U} \qquad \alpha g\left(T - T_0\right)\cos\phi \sim \delta \qquad \nu\nabla^2 u_\phi \sim -\nu\frac{U}{\lambda^2} \qquad \cancel{(\mathbf{u}\cdot\nabla)u_\phi \sim \frac{U^2}{L}}$$

Spatial averaging $\quad \langle\cdots\rangle = \dfrac{4}{\pi D^2 L}\displaystyle\int_0^{2\pi}\int_{-L/2}^{L/2}\int_0^{D/2}\cdots\; r\,dr\,dz\,d\theta$

Brown and Ahlers (PRL 2007, JFM 2008)

$$\boxed{\dot{\xi} = \beta_1 \xi - \beta_2 \xi^{3/2} + f_\xi(t)}$$

$\xi = \delta / \delta_0$; $\quad t \to t/\tau_\delta \quad \delta_0 = \dfrac{\nu^2 \, \mathrm{Re}^{3/2}}{\alpha g L^3}$

$\langle f_\xi(t) \rangle = 0; \quad \langle f_\xi(t) f_\xi(t') \rangle = 2\tilde{D}\delta(t - t')$ $\quad \beta_1, \beta_2 = O(1)$ geometrical prefactors

phenomenologically represents action of small-scale turbulent fluctuations on LSC

Navier-Stokes equation for u_θ : $\quad \dfrac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta = \nu \nabla^2 u_\theta$

$u_\theta \sim L\dot{\theta}$ LSC plane undergoes constant meandering as a rotating rigid body

$(\mathbf{u} \cdot \nabla) u_\theta \sim U\dot{\theta} \sim \delta\dot{\theta} \qquad \nu \nabla^2 u_\theta \sim -\nu \dfrac{L\dot{\theta}}{\lambda_\theta^2} \qquad \cancel{\langle \nu \nabla^2 u_\theta \rangle} \ll \langle (\mathbf{u} \cdot \nabla) u_\theta \rangle$

$\tau_\theta / \tau_\delta \ll 1$

We obtain: $\quad \boxed{\ddot{\theta} = -\gamma_1 \xi \dot{\theta} + f_{\dot{\theta}}(t)} \qquad \gamma_1 = O(1)$ geometrical prefactor

Brown and Ahlers (2007, 2008)

LSC amplitude probability distribution function

We have arrived at two coupled Langevin equations

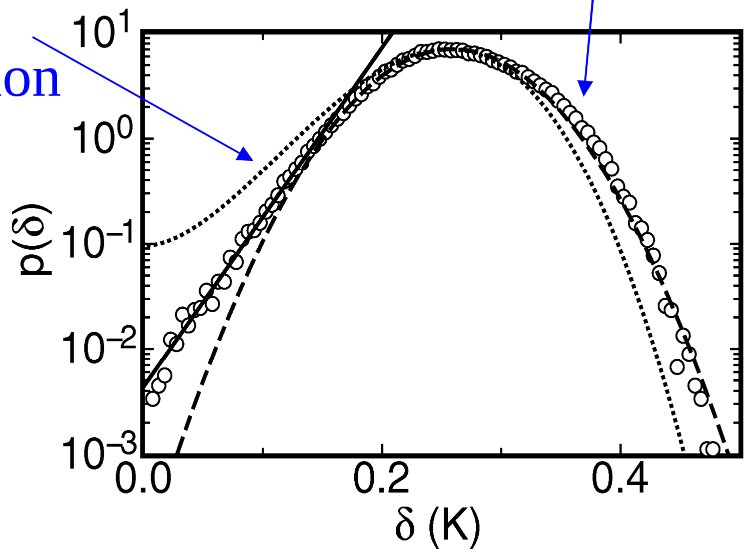
$$\dot{\xi} = \beta_1 \xi - \beta_2 \xi^{3/2} + f_\xi(t), \qquad \ddot{\theta} = -\dot{\theta}\gamma_1 \xi + f_{\dot{\theta}}(t)$$

Brown and Ahlers (2007, 2008)

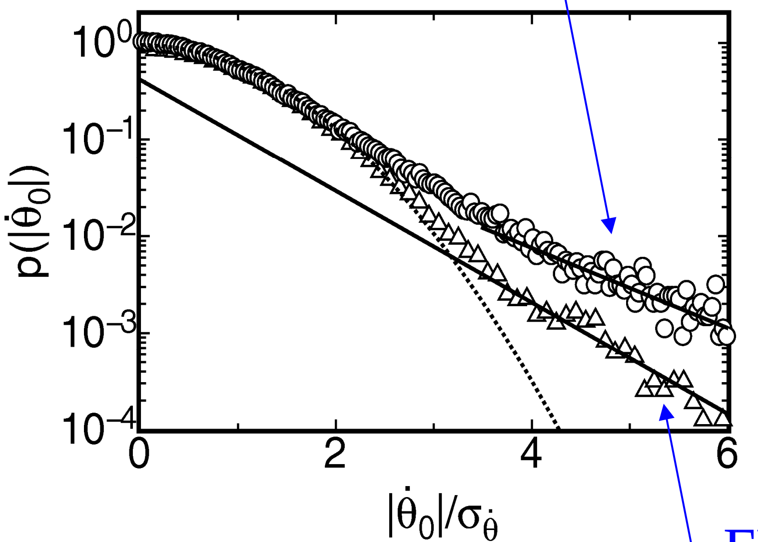
These predict well the typical behavior. However the tails are missed

FP solution

Experimental results



Experimental results



FP solution

Brown and Ahlers, JFM (2008)

LSC amplitude probability distribution function

To describe rare events, one has to correct both equations

$$\dot{\xi} = A + \beta_1 \xi - \beta_2 \xi^{3/2} + f_{\xi}(t)$$ at $\xi \to 0, \quad \dot{\xi} \sim Ra^{5/4}$ (Brown and Ahlers 2006)

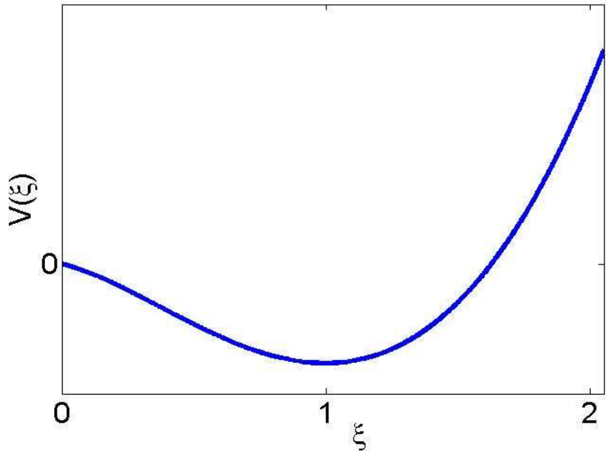
$$\ddot{\theta} = -\gamma_1 \xi \dot{\theta} - \gamma_2 \frac{\tau_\theta}{\tau_\delta} \xi^{1/2} \dot{\theta} + f_{\dot{\theta}}(t)$$ at $\xi << \sqrt{\tau_\theta / \tau_\delta}$ diffusion term dominant

These terms, negligible for $\xi \approx 1$ **become dominant** at cessation when $\xi \to 0$

First equation is decoupled

Corresponding Fokker-Planck equation is

$$\frac{\partial P(\xi,t)}{\partial t} = -\frac{\partial}{\partial \xi}\left[\left(A + \beta_1 \xi - \beta_2 \xi^{3/2}\right) P(\xi,t)\right] + \frac{\tilde{D}}{2} \frac{\partial^2 P(\xi,t)}{\partial \xi^2}$$

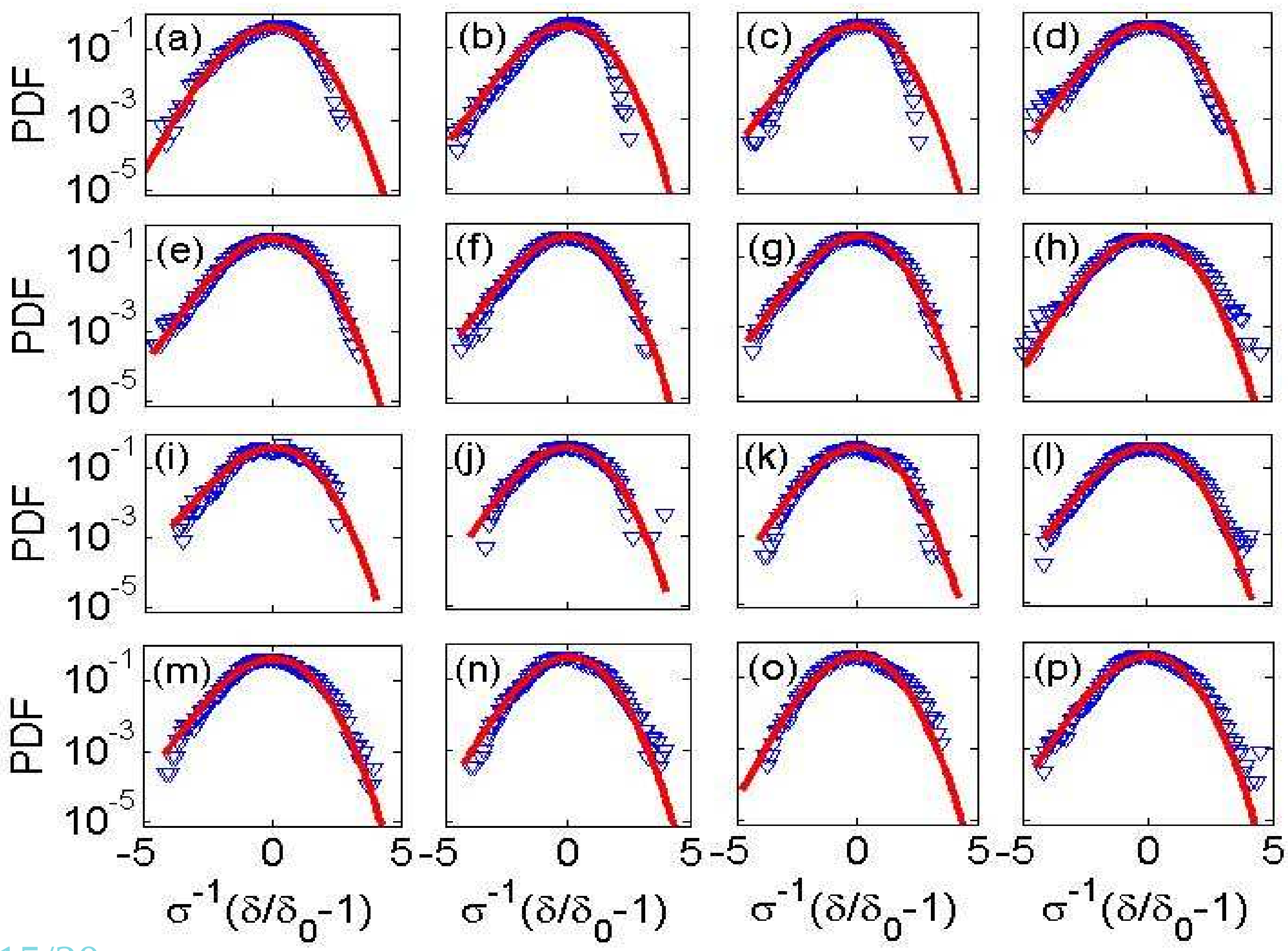


$$P(\xi,t) \sim e^{-2V(\xi)/\tilde{D}};$$

$$V(\xi) = -\int \left(A + \beta_1 \xi - \beta_2 \xi^{3/2}\right) d\xi = -A\xi - \frac{\beta_1 \xi^2}{2} + \frac{2\beta_2 \xi^{5/2}}{5}$$

Demanding that the PDF is centered about $\xi = 1$ with width $\tilde{D}$, we obtain

$$P(\xi) = \frac{1}{\sqrt{2\pi\tilde{D}}} \exp\left(-\frac{3B}{10} - \frac{1}{5\tilde{D}}\right) \exp\left\{B\xi + \frac{1}{\tilde{D}}\left[\left(1 - \frac{3B\tilde{D}}{2}\right)\xi^2 - \frac{4}{5}\left(1 - B\tilde{D}\right)\xi^{5/2}\right]\right\}$$



Left tail slope

$B = 2A/\tilde{D}$

$10^9 < Ra < 10^{11}$

Assaf, Angheluta and Goldenfeld (2011)

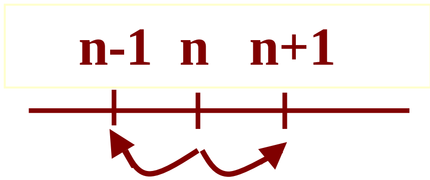
Cessation statistics

Cessation frequency is found by solving a first-passage problem:

Starting from the (vicinity of the) fixed point, what is the mean time it takes the system to reach the state $\xi = 0$?

Example: first passage time of random walker

Consider lattice sites: 0,…,N; 0 is absorbing, N is reflecting



$$T_n = \frac{1}{2}T_{n+1} + \frac{1}{2}T_{n-1} + 1 \qquad T_0 = 0, \qquad T_N = T_{N-1} + 1$$

$$U_n = T_{n+1} - T_n \quad \Rightarrow \quad \frac{1}{2}U_n - \frac{1}{2}U_{n-1} = -1 \quad \Rightarrow \quad U_n = U_{n-1} - 2 = U_{n-2} - 4 = \ldots = U_0 - 2n$$

$$T_n = T_0 + \sum_{i=0}^{n-1} U_i = n(1 - n + U_0) \qquad \Rightarrow \qquad U_0 = 2N - 1$$

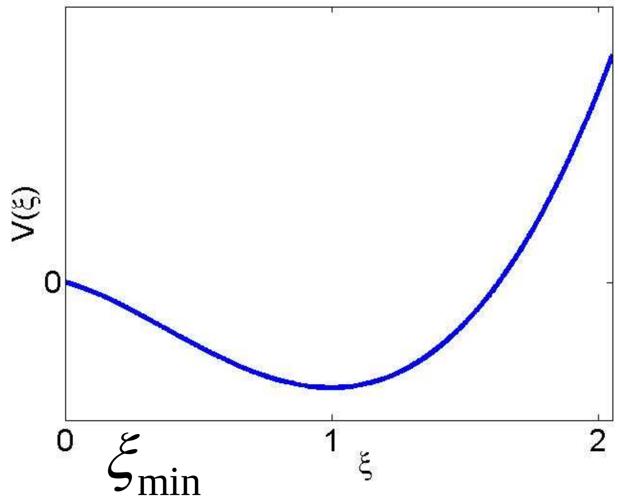
$$\boxed{T_n = n(2N - n)}$$

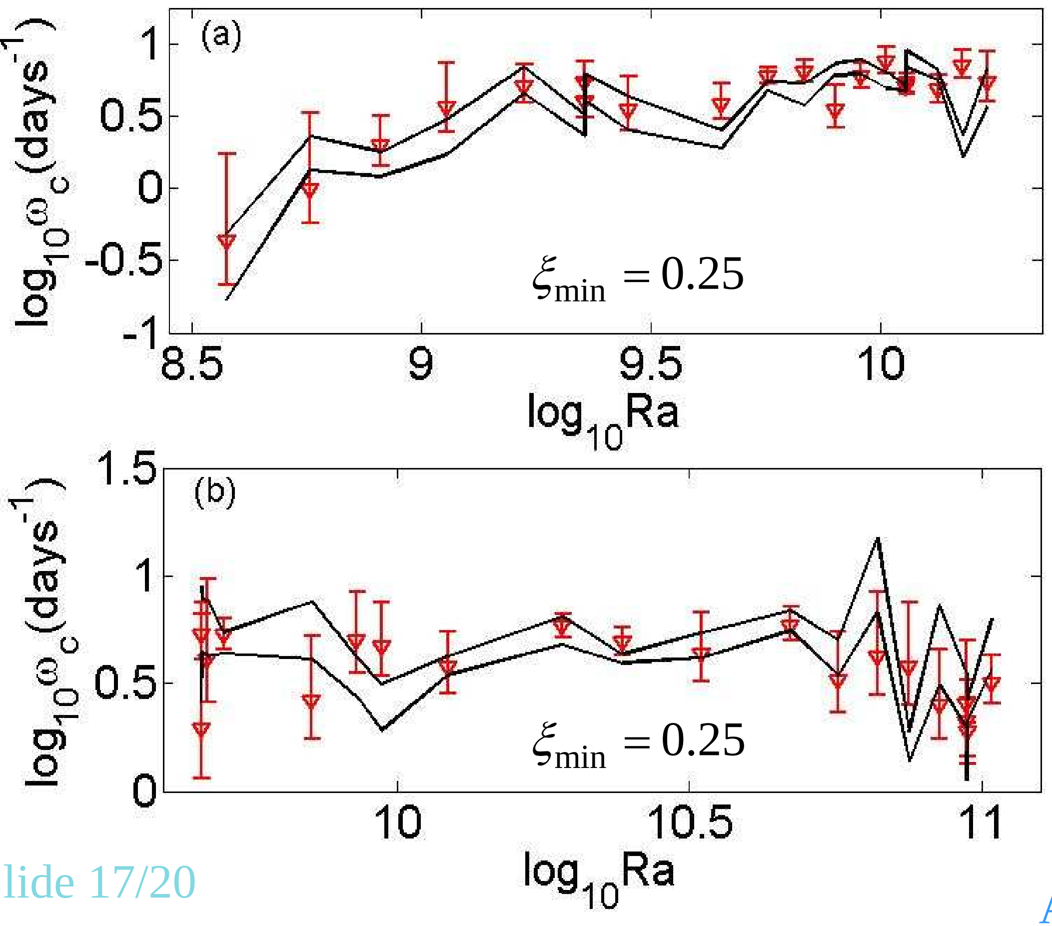
The mean time it takes "particle" to escape over potential barrier (in analogy to Kramers theory)

$$T_c(\xi_0) \approx \frac{\tau_\delta}{B}\sqrt{\frac{2\pi}{\tilde{D}}}\, e^{2\tilde{D}^{-1}[V(\xi_0)-V(1)]}$$

$$\psi(z) = e^{-2\int_{\xi_0}^{z} V(x)\,dx}$$

$$\xi_0 << 1$$





In experiment one assumes cessation occurs when $\xi < \xi_{\min}$

$$\boxed{T_c(\xi_{\min}) \approx \frac{1}{\xi_{\min}} \int_0^{\xi_{\min}} T_c(\xi_0)\,d\xi_0}$$

$\omega_c = T_c^{-1} =$ cessation frequency

Assaf, Angheluta and Goldenfeld (2011)

Angular velocity probability distribution function

$$\ddot{\theta} = -\dot{\theta}\left(\gamma_1\xi + \gamma_2 \frac{\tau_\theta}{\tau_\delta}\xi^{1/2}\right) + f_{\dot{\theta}}(t)$$

$$\boxed{P(\Delta\theta) \sim P(\dot{\theta}) = \int_0^\infty P(\dot{\theta}\,|\,\xi)P(\xi)d\xi}$$

valid for $\tau_\theta / \tau_\delta << 1$, namely if $P(\dot{\theta}\,|\,\xi)$ equilibrates much faster than the typical timescale of change of ξ

$$P(\dot{\theta}\,|\,\xi) = \frac{1}{\sqrt{2\pi D_{\dot{\theta}}}} \exp\left[-\frac{1}{D_{\dot{\theta}}}\left(\gamma_1\xi + \gamma_2\frac{\tau_\theta}{\tau_\delta}\xi^{1/2}\right)\dot{\theta}^2\right]$$

$$P(\xi) \sim \exp\left\{B\xi + \frac{1}{\tilde{D}}\left[\left(1 - \frac{3B\tilde{D}}{2}\right)\xi^2 - \frac{4}{5}\left(1 - B\tilde{D}\right)\xi^{5/2}\right]\right\}$$

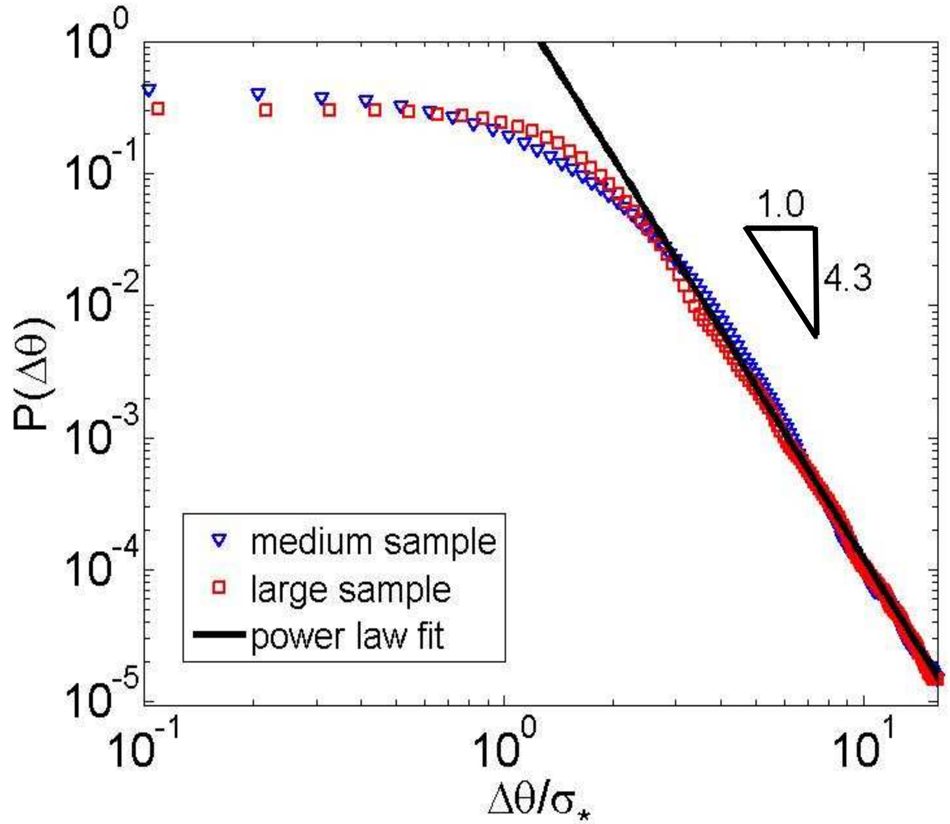
In Gaussian regime, one puts $\xi = 1$ to obtain $P(\dot{\theta}) \sim e^{-\dot{\theta}^2 / D_{\dot{\theta}}}$

What is the probability for large $\Delta\theta$?

when cessation occurs it is easier for system to undergo large changes in θ

Right tail of $P(\Delta\theta)$ **dominated by** $\xi \ll 1$

$$P(\dot{\theta}|\xi) \sim \exp\left(-\frac{\gamma_2}{D_{\dot{\theta}}}\frac{\tau_\theta}{\tau_\delta}\xi^{1/2}\dot{\theta}^2\right)$$



$$P(\Delta\theta)_{\frac{\Delta\theta}{\sigma_*}>1} \sim \frac{2\gamma_2}{\sqrt{\tau_\delta/\tau_\theta}} D_{\dot{\theta}}^2 \left(\Delta\theta\right)^{-4}$$

without the term $\sqrt{\xi}$

the tail scales as $(\Delta\theta)^{-2}$

Assaf, Angheluta and Goldenfeld (2011)

Conclusions

- Cessations of LSCs are rare and striking phenomena. Accurately assessing their probabilities is of great importance in many applications

- We have investigated LSC in the realm of RBC: it includes all physical ingredients found in natural LSCs and cessations, and is well-controlled

- We have extended a stochastic model describing LSC dynamics in turbulent RBC, to account for rare events. This was done by including terms negligible in the "typical behavior" regime but important in the cessation regime

- Our results agree excellently with careful analysis of experimental data

Thank you!